From hyperbolic 2-space to euclidean 3-space

Tilings and patterns via topology

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The Gauss Map of a surface patch, S

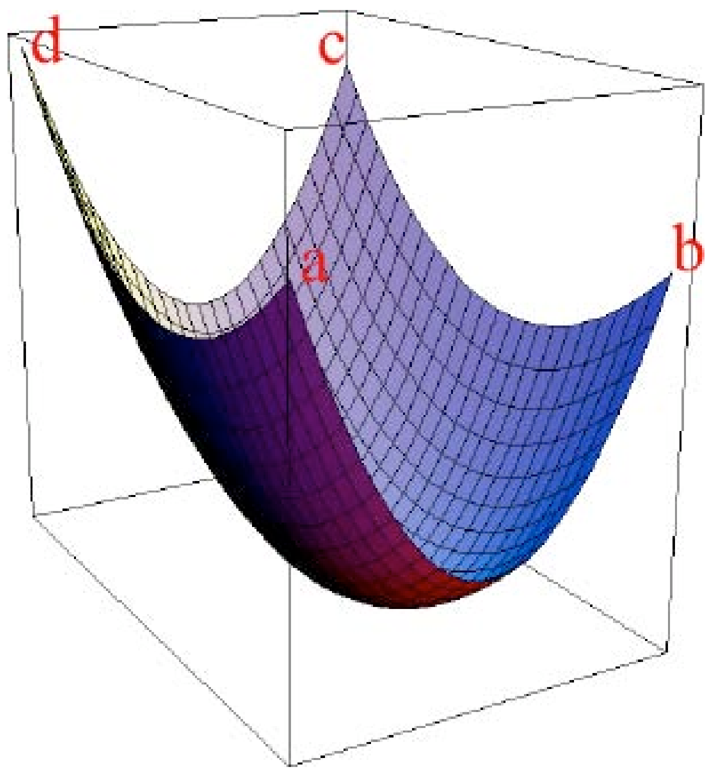
- determine an outward-facing unit normal vector at $x \in S$: $\mathbf{n}(\mathbf{x})$.

- move the Gauss sphere so that it is centred at x.

- The Gauss map of x is the endpoint of $\mathbf{n}(\mathbf{x})$ on the Gauss sphere.

- Do this for all points on the surface patch, S

- resulting patch on the Gauss sphere is the Gauss map of S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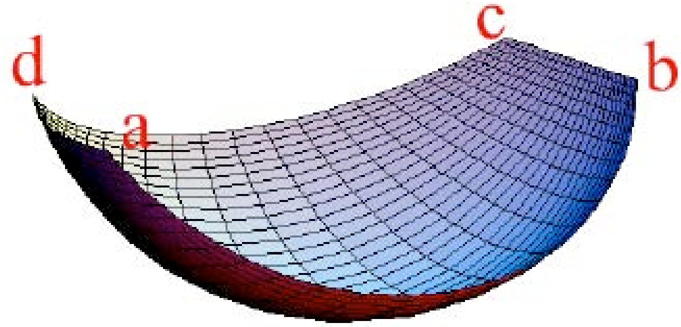
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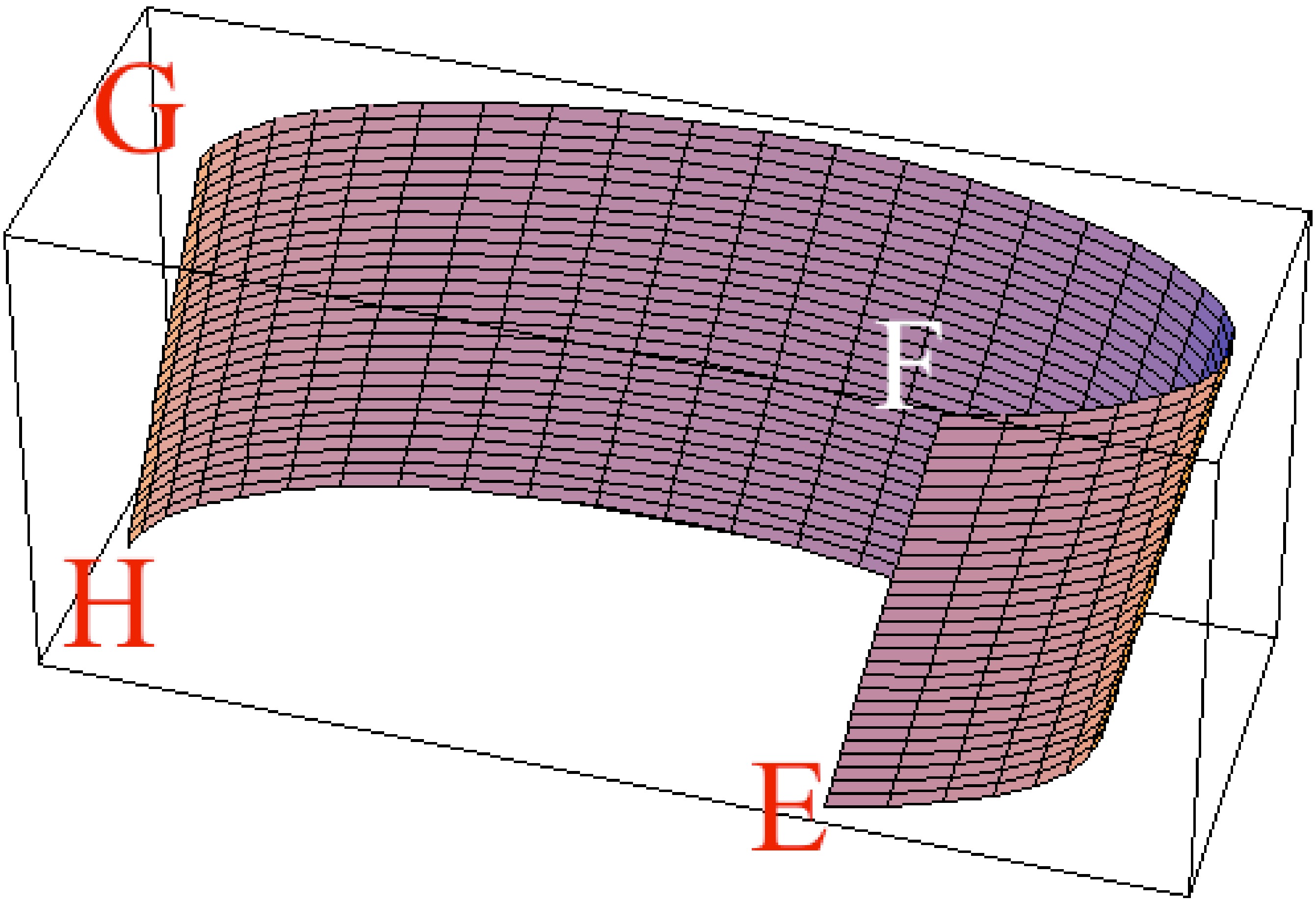
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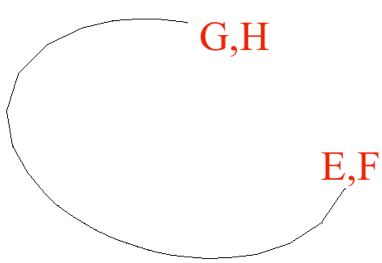
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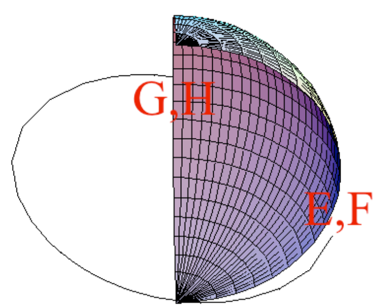
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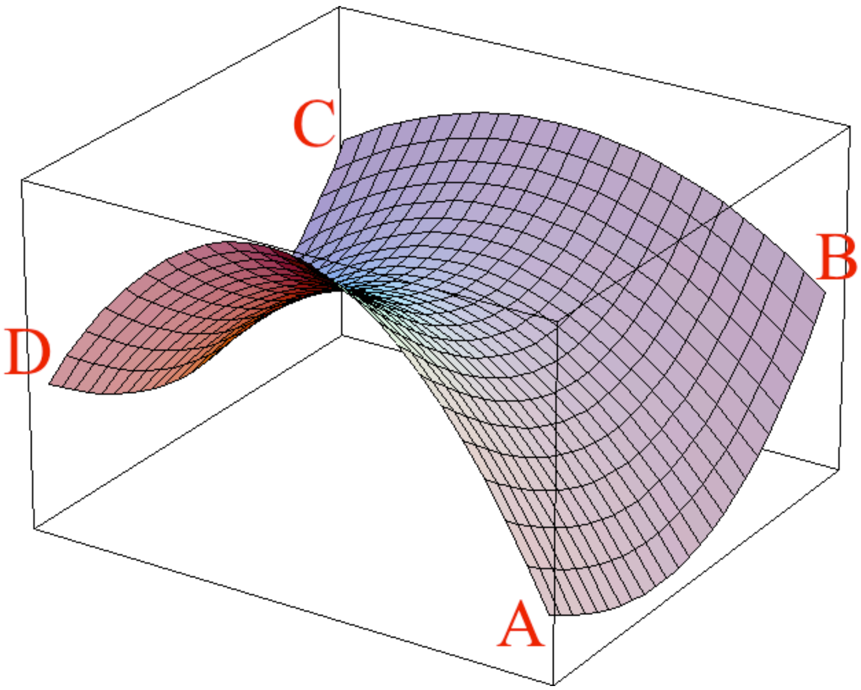
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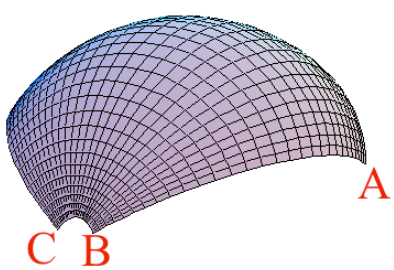
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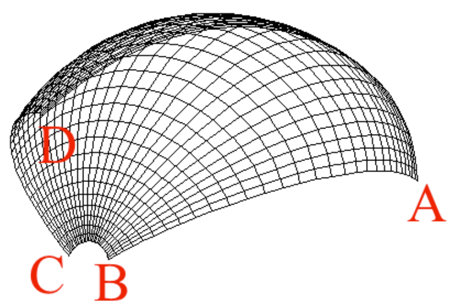
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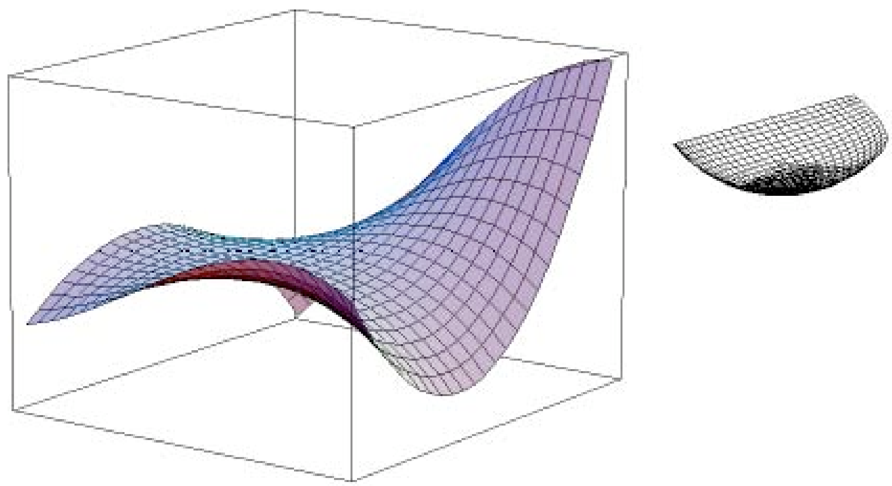

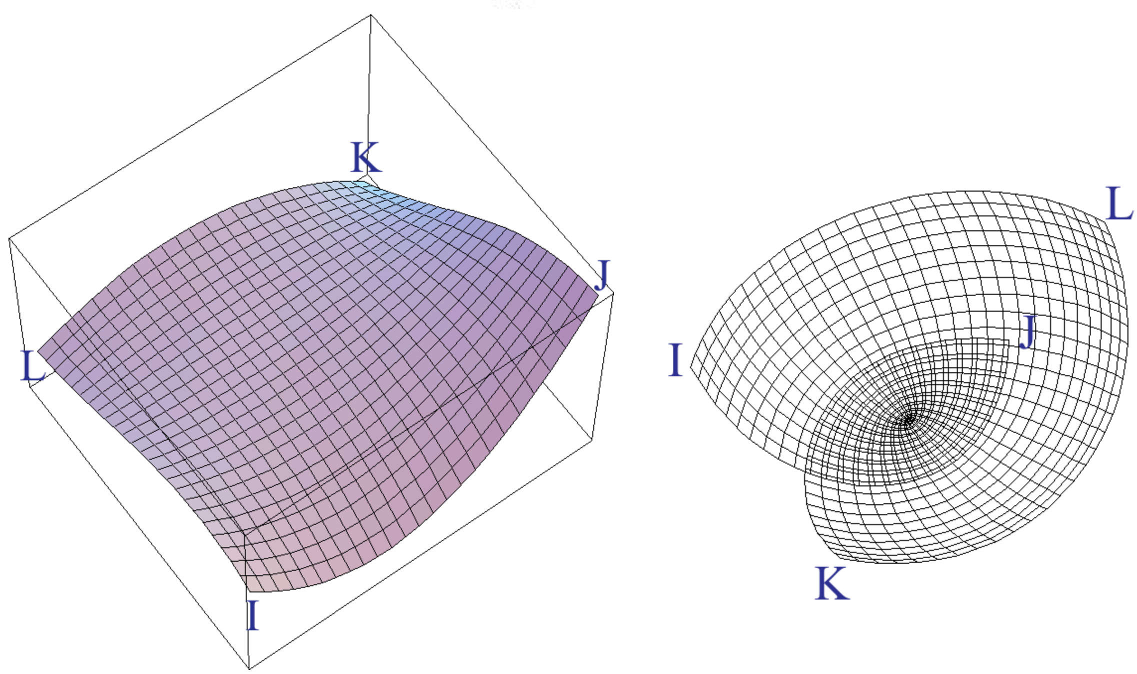
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Gaussian curvature (generic solid angle)

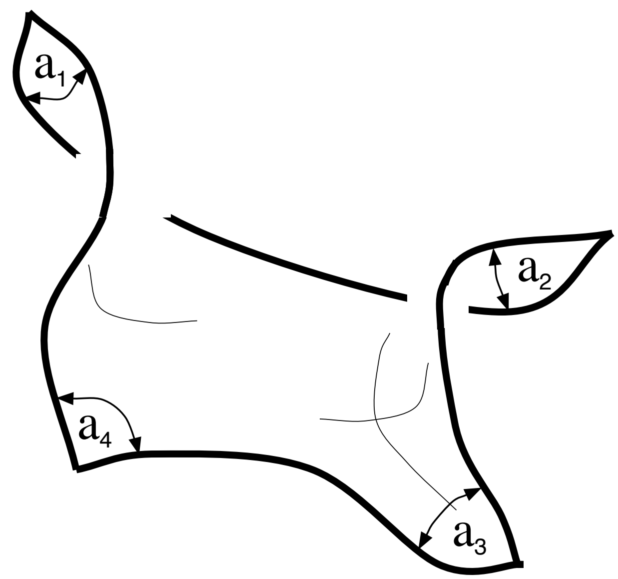
The area of S on the Gauss sphere ($G(S)$) is the **integral curvature** (of the surface patch S).

if $K(x)$:= Gaussian curvature at x:

$$\int\int_S |K| da = G(S)$$

Local Gauss-Bonnet formula

Suppose the patch S is bounded by a geodesic polygon containing v vertices, with interior angles between edges at each vertex α_j:



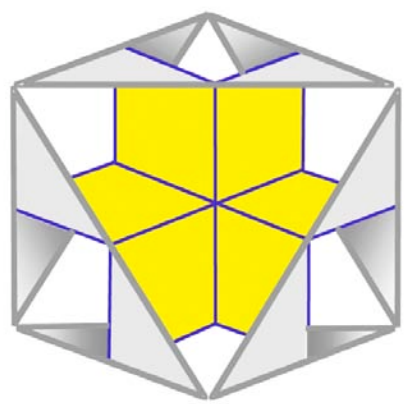
$$\int\int_S K da = (2-v)\pi + \sum_{j=1}^{v} \alpha_j$$

Exercise 1 *Draw the Gauss maps of*

1. *an upper hemisphere*

2. *a region surrounding a cube vertex*

3. *a region surrounding a vertex of the tiling with Schläfli symbol* $\{6,4\}$*:*



and determine their total areas on the sphere, given that the area of a single cover of the Gauss sphere is 4π*.*

4. *Form the Gauss map of a cone with apex angle* a*, giving a sector of interior angle* 2β*. Determine the integral curvature as a function of* a*. Unzip the cone to form a pie-shaped sector in the plane. What is the relationship between the apex angle* a *and the sector angle? Write down the integral curvature for a sector angle of* $\frac{2\pi}{k}$*.*

SOLUTIONS to EXERCISES 1:

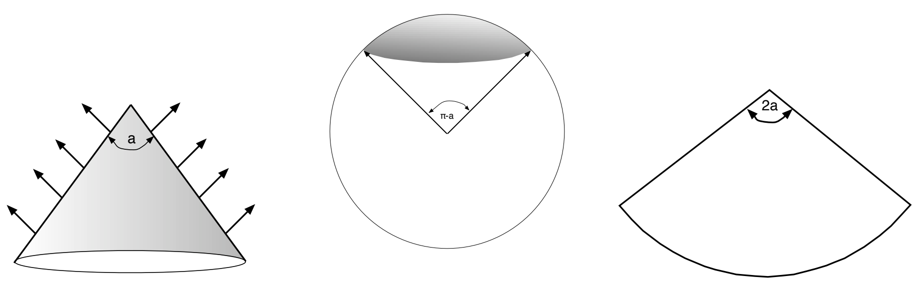
1.

2.

3.

4.

5. The Gauss map is a cap of the sphere, subtending an angle at the centre of $\pi - a$, giving a total area (or solid angle) of $\frac{2\pi(\pi-a)}{\pi} = 2(\pi - a)$.

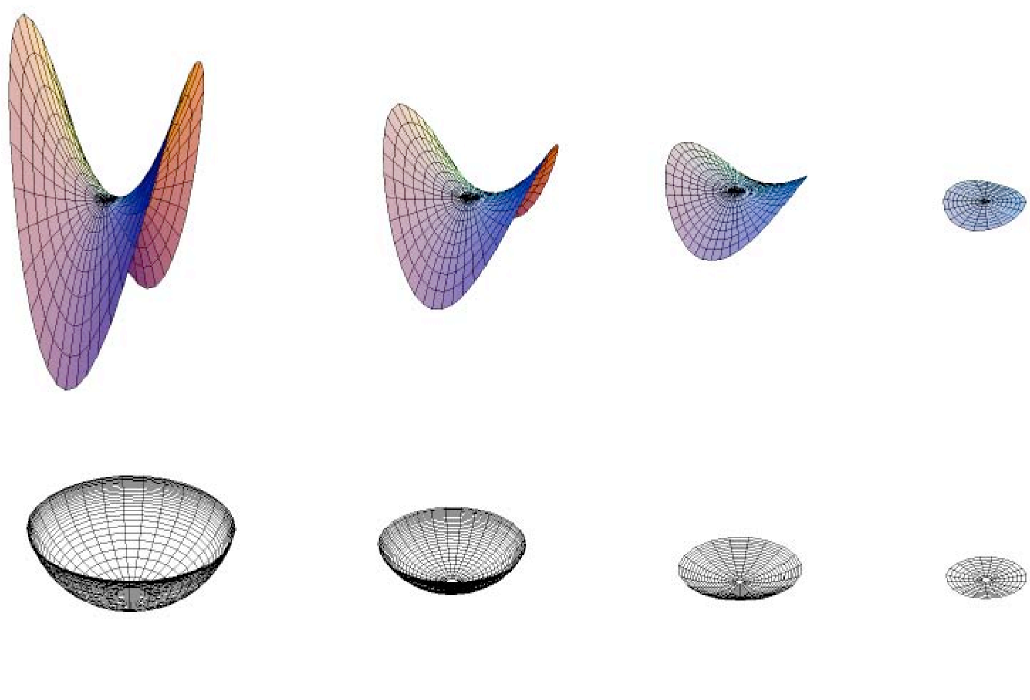


The unzipping procedure gives a sector with angle $2a$. If $2a = \frac{2\pi}{k}$, the integral curvature is $2(\pi - a) = 2\pi - \frac{2\pi}{k} = \frac{2\pi(1-k)}{k}$.

Gaussian curvature, K

The Gaussian curvature, K, at a point on the surface, x can be written in terms of the area of the Gauss map, $G(S)$ of the surface patch, S as it shrinks to x:

$$\lim_{S \to x} \frac{G(S)}{S}$$

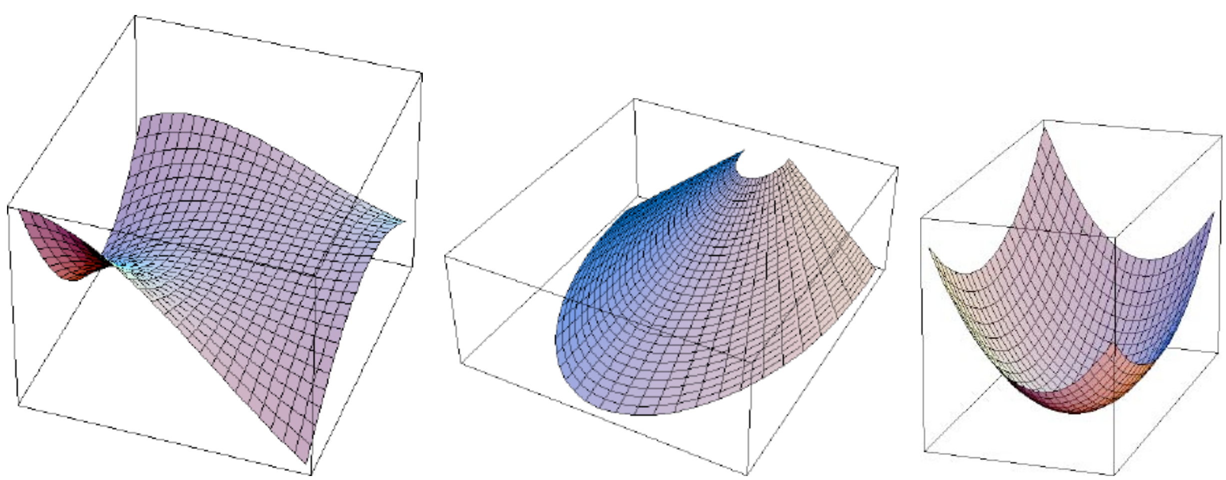


- Gaussian curvature does not change we bend or fold a surface, provided the surface is not stretched or compressed!

- Cylinder, cone, plane: $K = 0$.

- Saddles have $K < 0$

- Caps have $K > 0$.



Exercise 2 *Determine the sign of the Gaussian curvature of the patches above.*

SOLUTION: $-, +, 0$

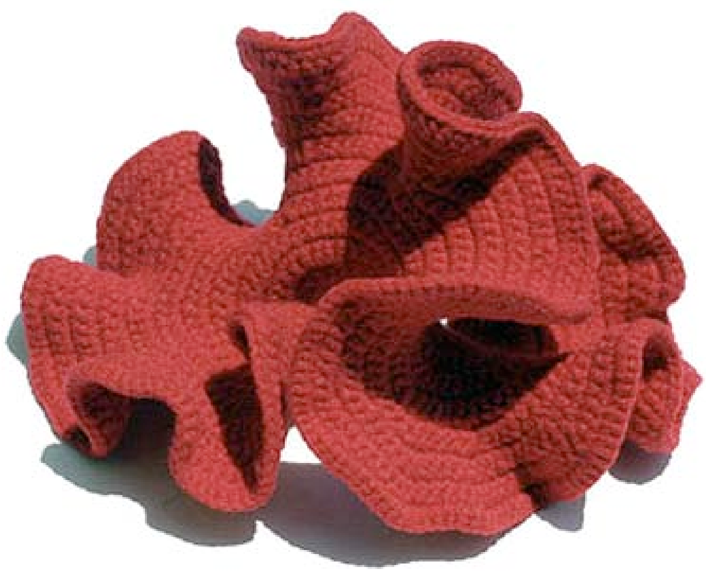
$2D$ Non-Euclidean Geometry

homogeneous surfaces have constant Gaussian curvature at *all* points on the surface

Define *spaces* by metrics inherited by those surfaces (independent of embeddings in higher space!) **three non-euclidean** $2D$ **spaces**:

- Euclidean plane, $\mathbf{E^2}$: $K = 0$

- **hyperbolic plane**, $\mathbf{H^2}$: $K = -1$



Daina Taimina

- elliptic plane $\mathbf{S^2}$: $K = +1$

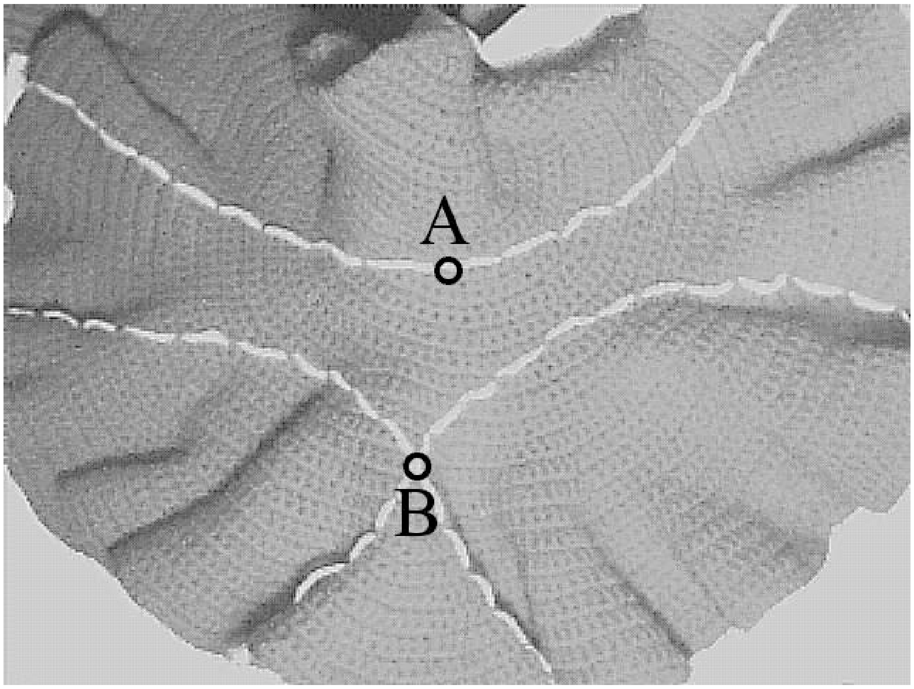
All extended, disc-like, simply-connected regions (without any internal holes or handles).

Comparing $2D$ elliptic ($\mathbf{S^2}$), euclidean ($\mathbf{E^2}$) & hyperbolic ($\mathbf{H^2}$) spaces:

Feature	$\mathbf{S^2}$	$\mathbf{E^2}$	$\mathbf{H^2}$
# parallel directions	0	1	many
triangle angle sum	$< \pi$	π	$> \pi$
area growth (wrt radius) R	$\sin^2(R)$	R^2	$\sinh^2(R)$ $\exp(R)$

Exponential growth of area in $\mathbf{H^2} \sim \infty-$dimensional euclidean space!

So $\mathbf{H^2}$ must multiply cover normal 3-space ($\mathbf{E^3}$).



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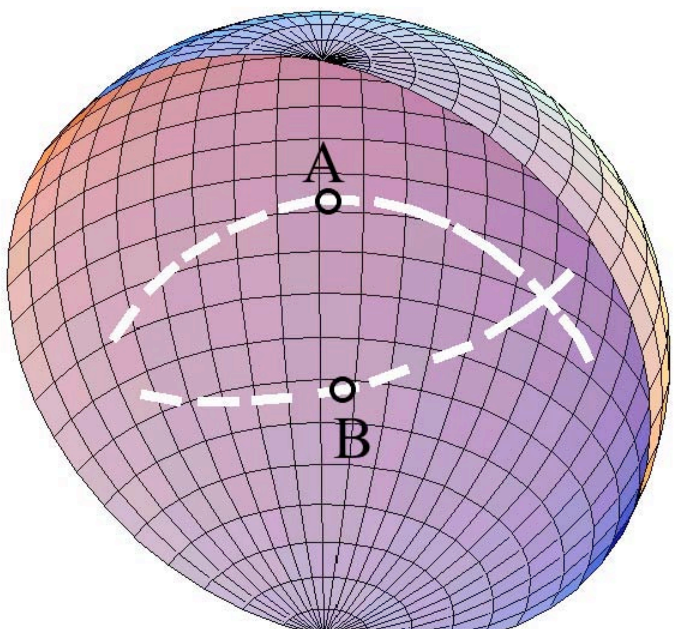
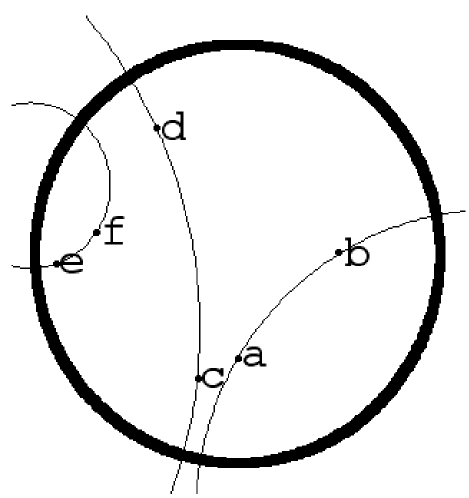


Parallel geodesics on embeddings in space of fragments of (top) $\mathbf{H}^2$ and (bottom) $\mathbf{S}^2$

Poincaré disc model of $\mathbf{H^2}$:

squeeze entire space into a euclidean unit disc

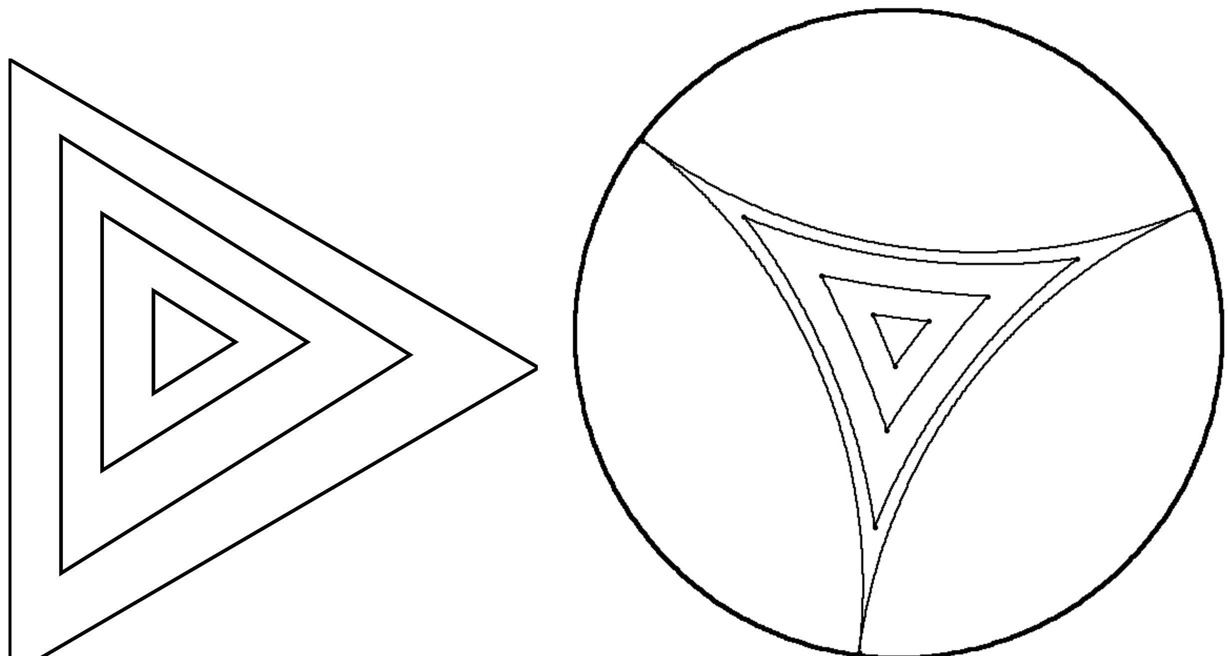
- geodesics in $\mathbf{H^2}$ are circular arcs that intersect the Poincaré disc boundary at right angles

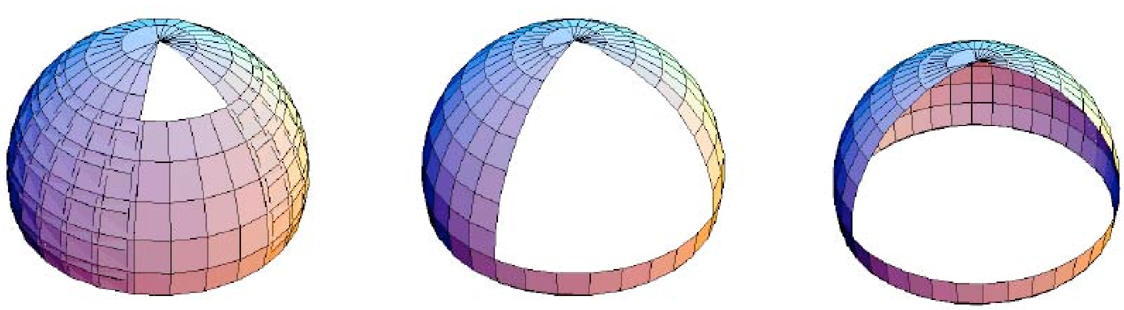


ab, cd and ef don't intersect – they are all (ultra)parallel geodesics in $\mathbf{H^2}$!

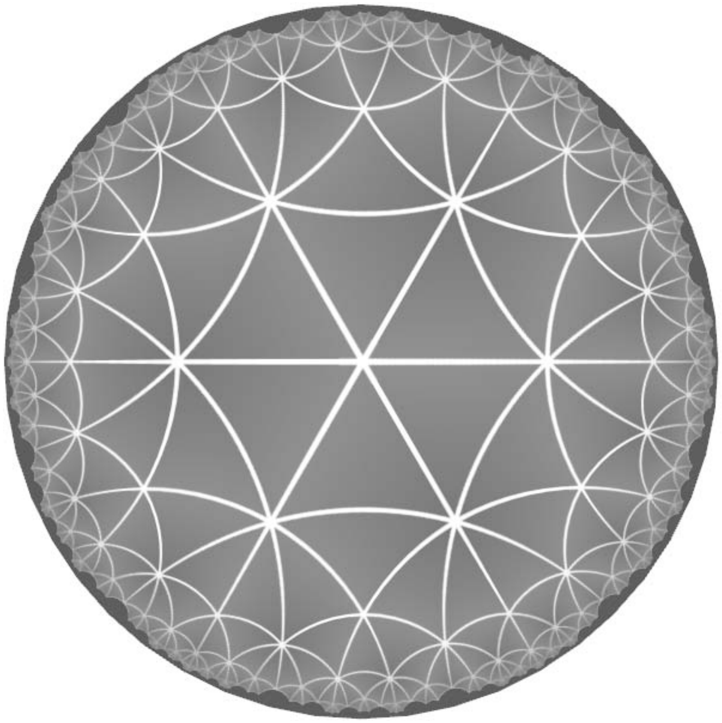
- shrinkage of actual lengths – increase as we move to disc boundary (∞ in $\mathbf{H^2}$)

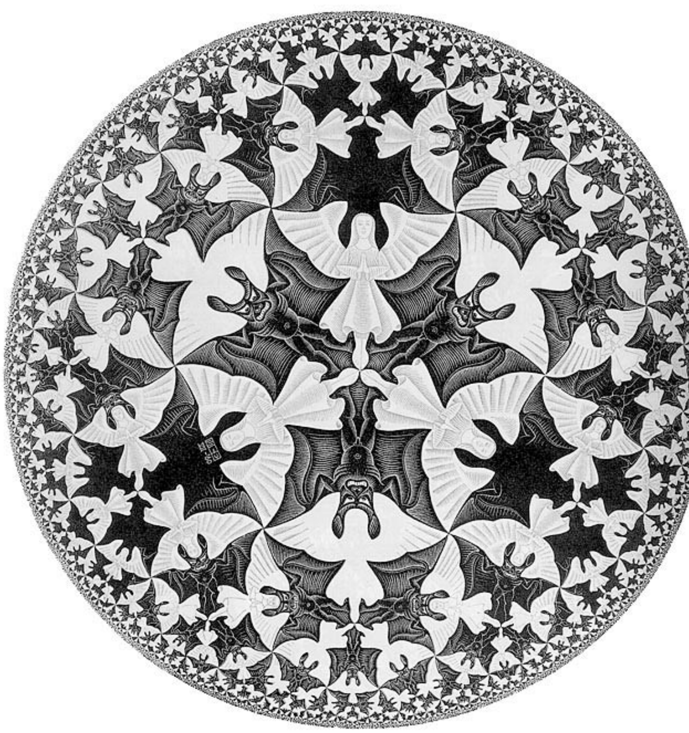
- all angles measured on disc are identical to true angles in $\mathbf{H}^2$ (conformal)





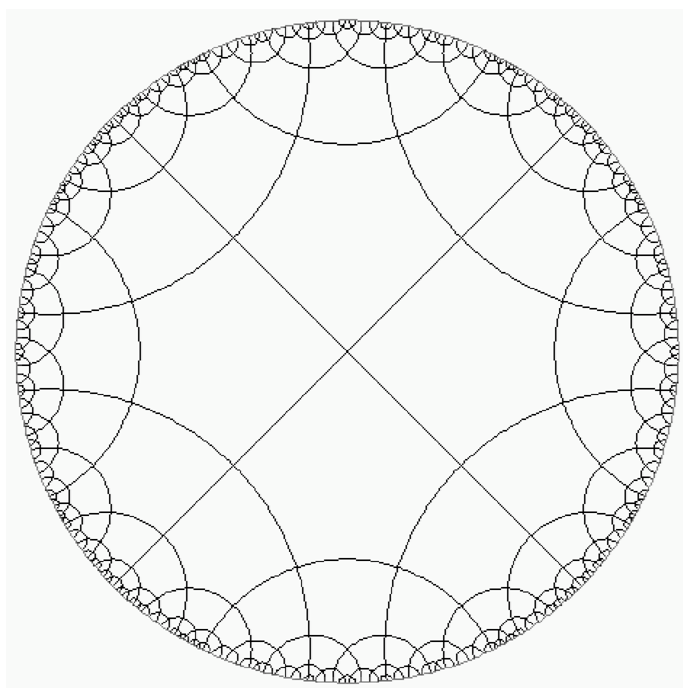
Triangles on all (Riemannian) $2D$ spaces are euclidean when small . . .





M.C. Escher. *Circle Limit IV: Angels and Devils*

$\boxed{\text{Poinc disc movie}}$

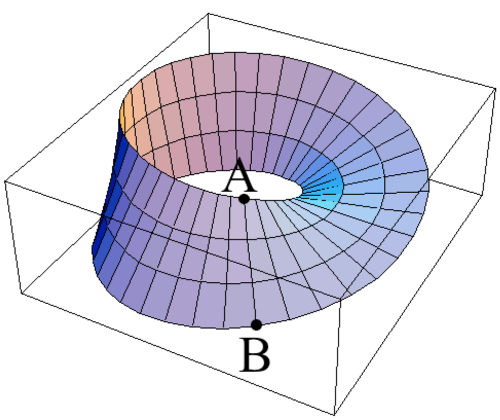


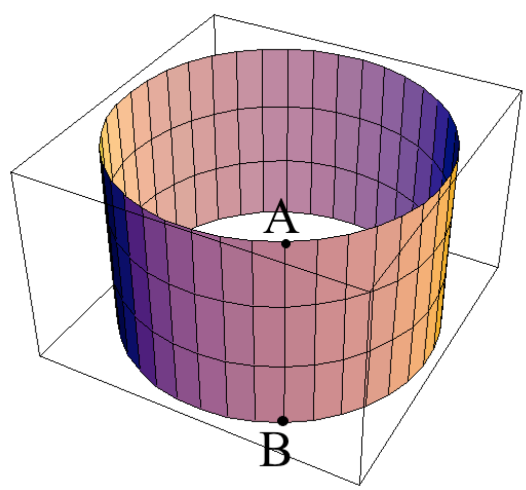
Topology of surface manifolds and orbifolds

Topology is 'rubber-sheet' geometry.
(e.g. all simple polyhedra are *homeomorphic* to a sphere)

- *First, distinguish* **orientable** *from* **non-orientable** *manifolds*

  Orientable manifolds allow for a normal pointing in one sense at each point, and no reversal by transport around the manifold.





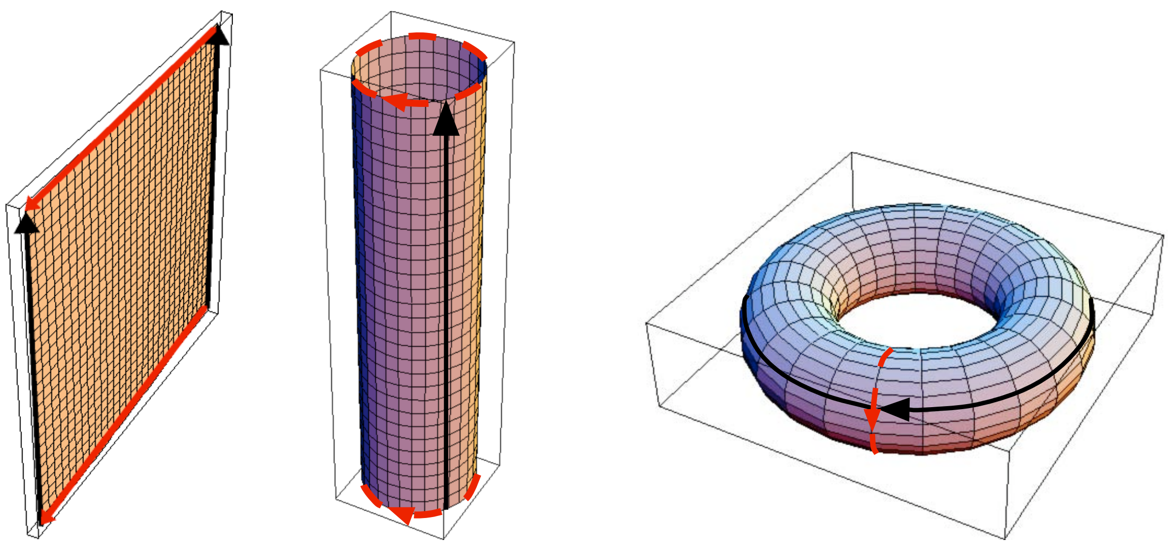
(Left) The non-orientable Moebius strip contains a single boundary loop (ABA), while its orientable analogue (right) contains a pair of disjoint boundary loops (AA and BB).

- *Second, distinguish* **boundary-free manifolds** *from* **manifolds with boundary** *(edge-free or containing edges)*

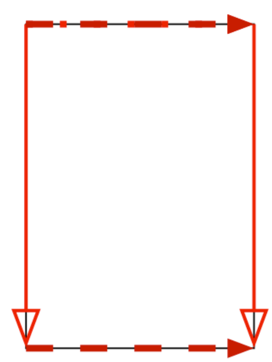
<u>Boundary-free manifolds</u> can have:

- **handles** if orientable (# handles = *oriented genus*)
- **cross-caps** if non-orientable (# cross-caps = *non-oriented genus*)

Handles: $\circ$



The *unzipped* form of the torus (a single boundary-free handle) is a 4-gon with 2 parallel zip-pairs:

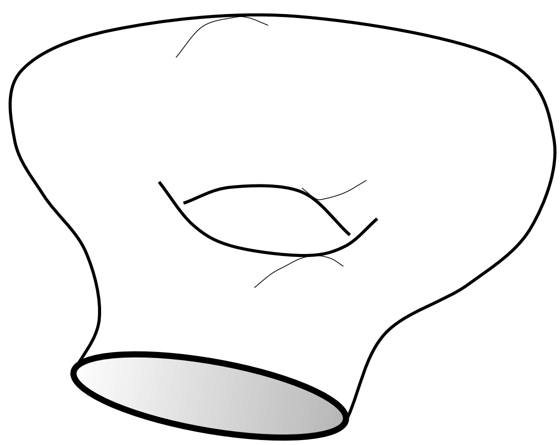


From the local Gauss-Bonnet formula:

$$\int\int_S K da = (2-4)\pi + \sum_{j=1}^{4} \pi/2 = 0$$

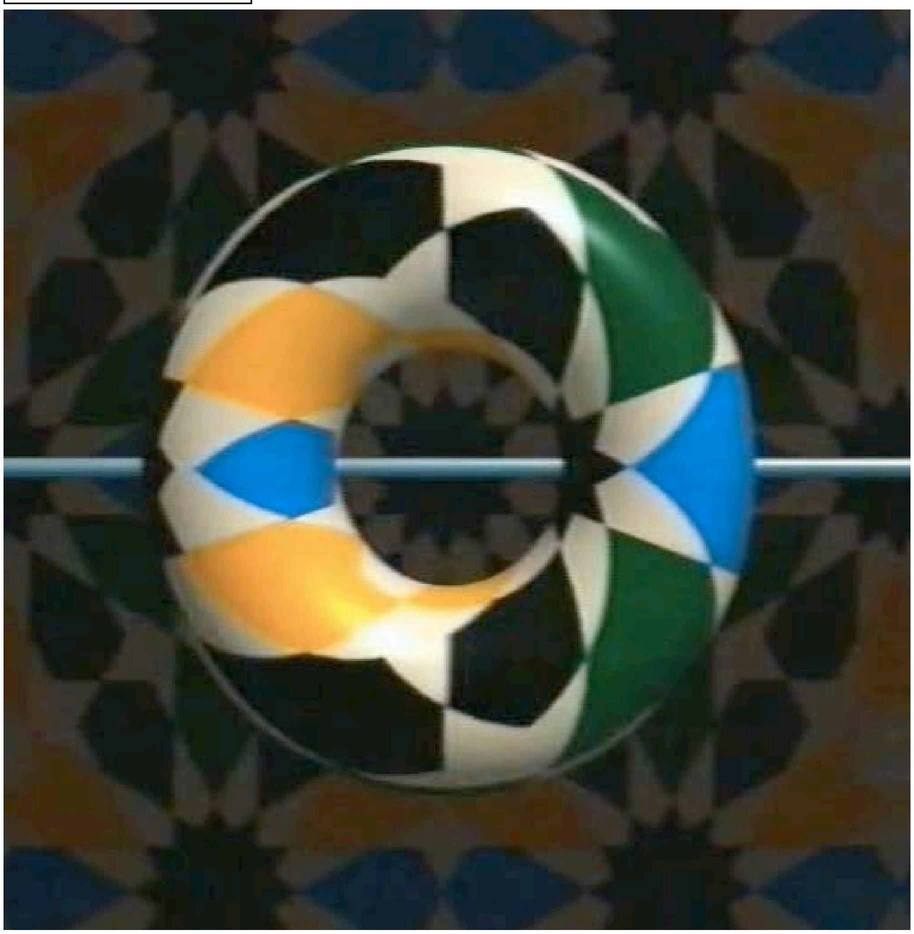
So, *all* tori are, at least on average, euclidean!

Form one half of a zip-pair on a torus by punching out a disc.



This is a handle $- \circ$.

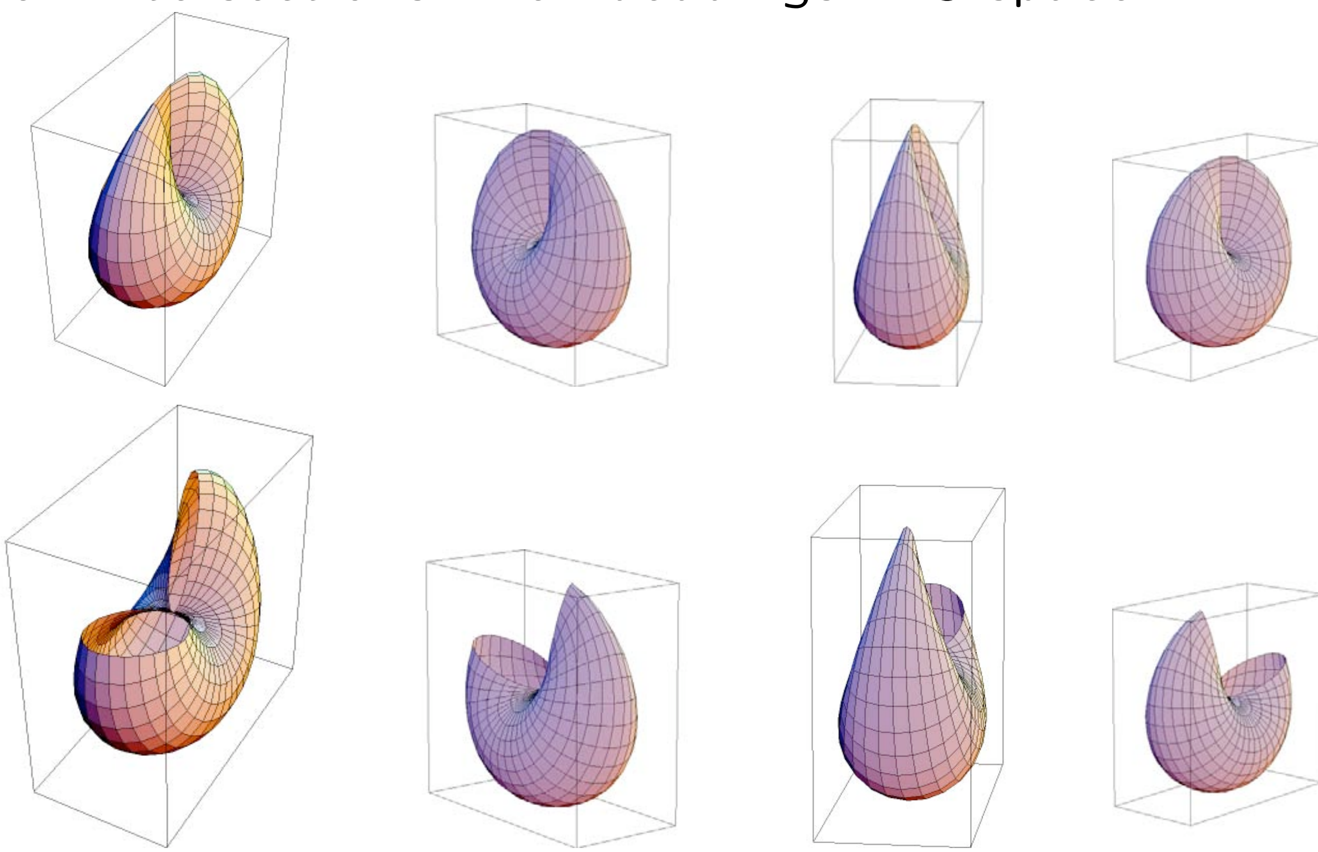
$\boxed{\text{torus movie}}$



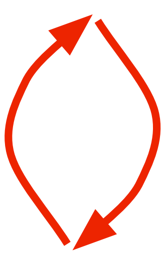
'Symmetry at the Alhambra', Antonio Costa and Bernardo Gómez, 1999

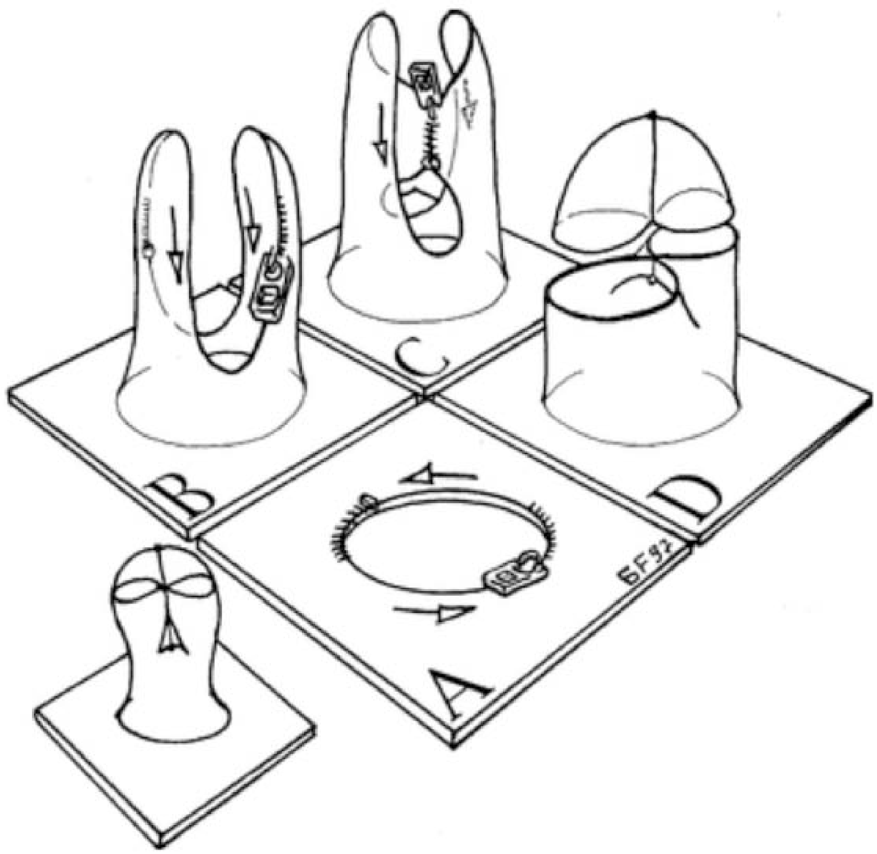
Cross-caps: $\times$

. . . difficult to visualise due to
self-intersections in embeddings in 3-space:



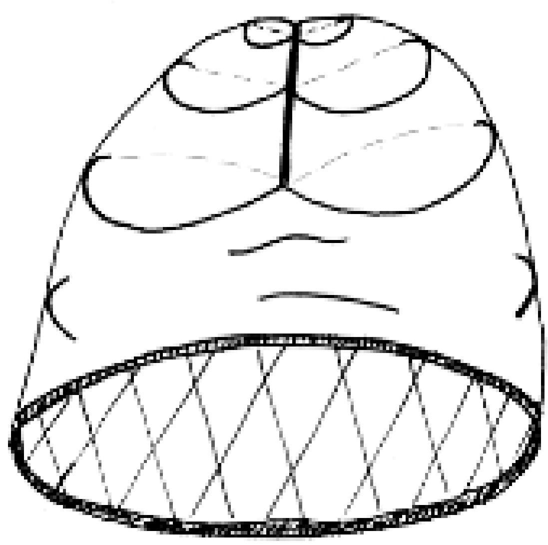
. . . construct by zipping a single zip-pair:





So the *unzipped* form of the cross-surface (a single boundary-free cross-cap) is a 2-gon with 1 anti-parallel zip-pair.

Form one half of a zip-pair on a cross-surface by punching out a disc.



This is a cross-cap – $\times$.

Exercise 3 *Is the cross-surface intrinsically elliptic, euclidean or hyperbolic?*

SOLUTIONS to to EXERCISE 3: Use the local Gauss-Bonnet theorem to deduce the integral curvature of any *cross-surface. Noting that its geodesic polygon contains two vertices gives:*

$$\int\int_S K da = (2-2)\pi + 2\pi = 2\pi$$

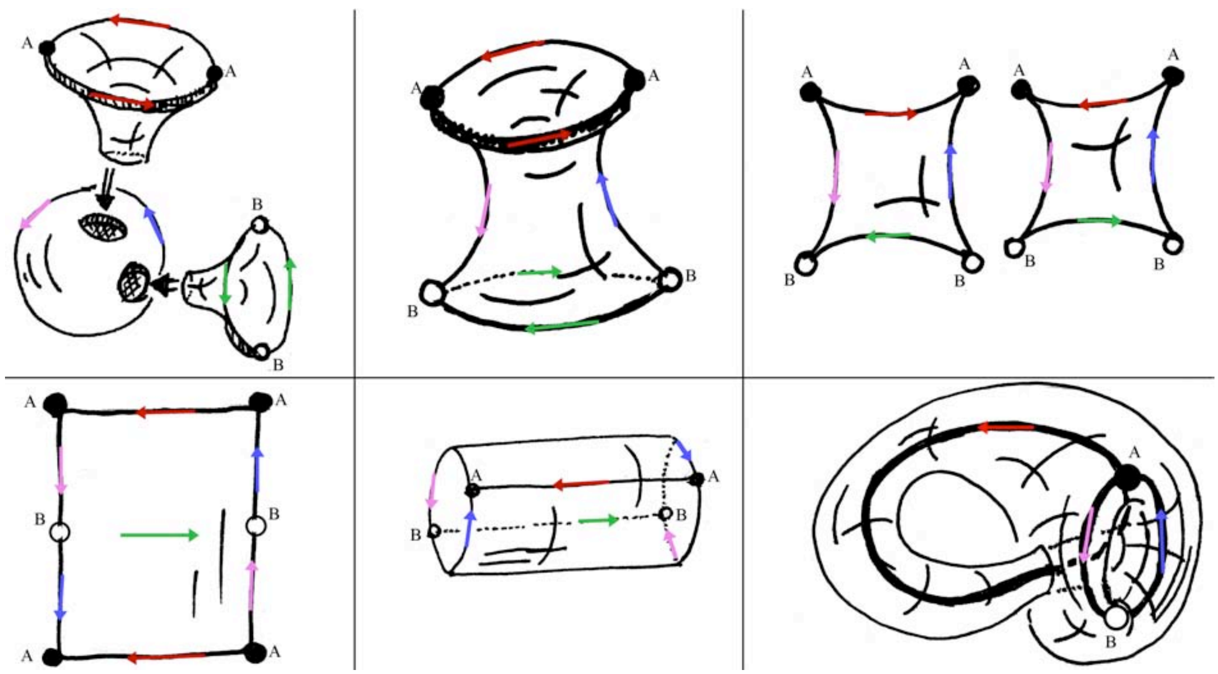
Therefore all cross-surfaces have positive integral curvature; in contrast to tori, their average geometry is elliptic.

Boundary-free 2-manifolds

zipped handles and cross-caps

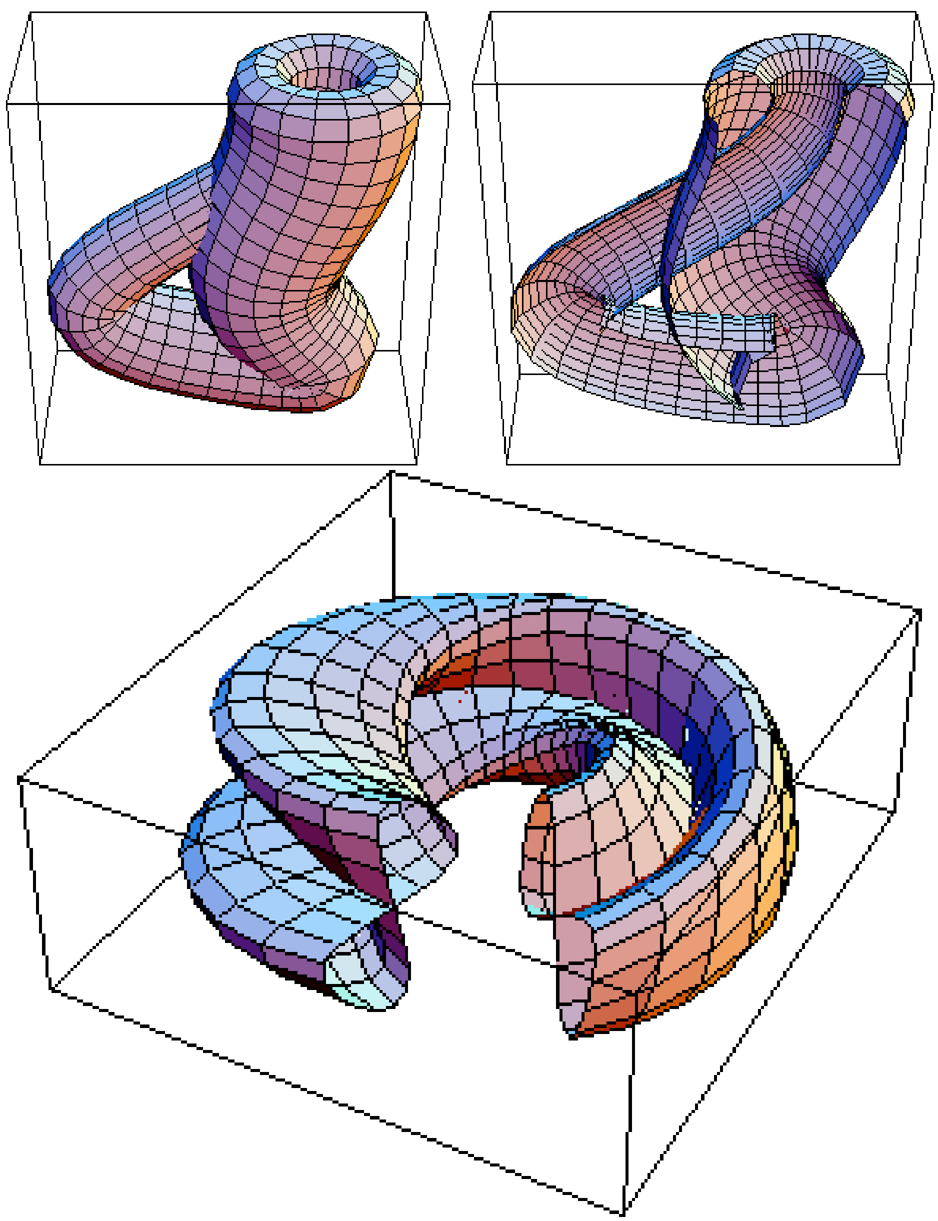
Use handle and cross-cap modules; zip into the sphere.

e.g. What manifold is produced by a pair of cross-caps, $\times\times$?

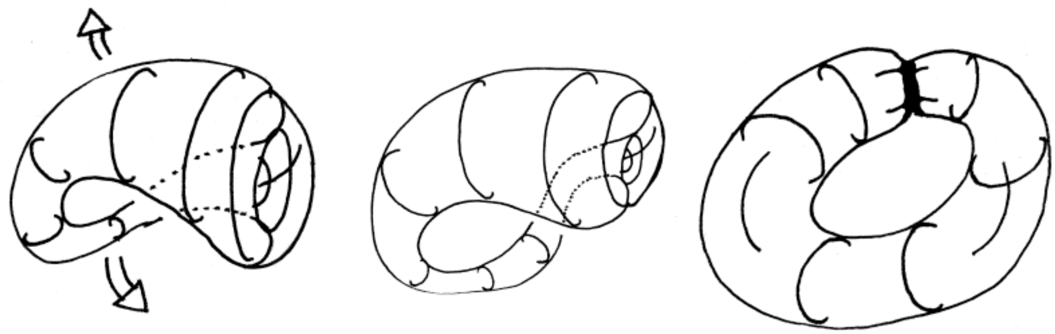


The *Klein bottle*.

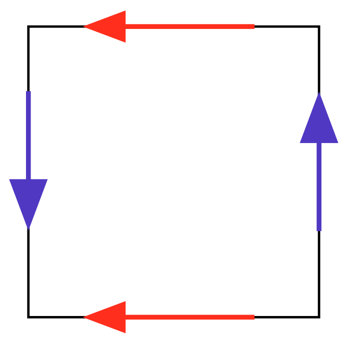
Many alternative embeddings of this manifold can be drawn:



Forming the *cross-handle*:



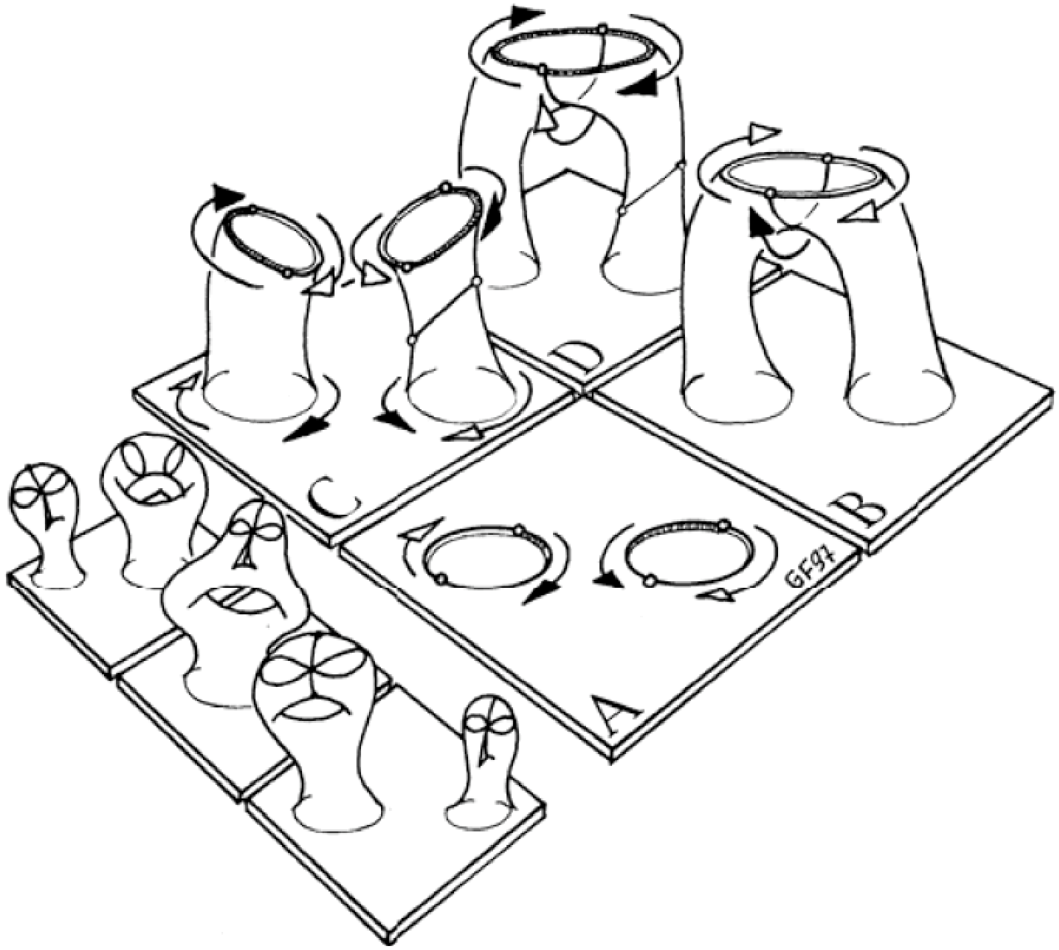
The unzipped form of this $\times\times$ manifold is a 4-gon, similar to the torus, $\circ$:



Both $\circ$ and $\times\times$ have the same average geometry (euclidean)!

But $\circ$ has an orientable handle (orientable genus 1), while $\times\times$ has a non-orientable cross-handle (non-orientable genus 2).

A single cross-cap can be moved through handles, leaving them as cross-handles:



Mixed $\circ, \times$ manifolds are invalid . . .

$$\circ^n \bigoplus \times = \times^{2n+1}$$

All boundary-free manifolds with genus, $g > 0$ contain

$$\underline{\circ\ldots}$$

OR

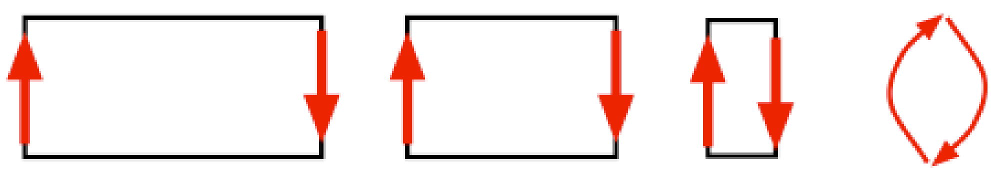
$$\underline{\times\ldots}$$

Bounded manifolds

with n separate boundary loops are formed by removing n discs

...append $\underline{* * \cdots *}$ to the symbol.

e.g. Puncturing the cross-surface, $\times$ gives:



. . . the Möbius strip,

$\times *$

All manifolds with genus > 0 contain

- $\circ \cdots * \ldots$

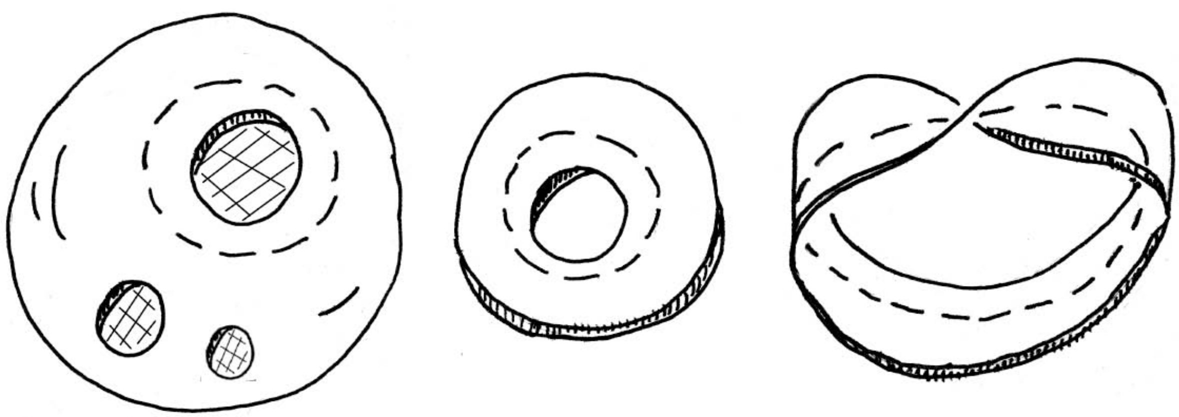
OR

- $\times \cdots * \ldots$

$\#$ of $\circ$ or $\times$ entries is the surface genus!

The sphere, the disc ($*$) and the cross-surface ($\times$) are *simply connected*

All other manifolds are *multiply connected*:



$* * *$, $**$ and $*\times$

dashed loops cannot shrink to a point in these manifolds, making them multiply-connected

Euler-Poincaré characteristic of manifolds, χ

• The *global Gauss Bonnet* formula gives the integral curvature of a boundary-free surface $\mathcal{S}$ as a topological constant, χ:

$$\int\int_{\mathcal{S}} K da = 2\pi\chi(\mathcal{S})$$

Use this to work out χ and the decrease of χ, $\delta\chi$ for standard surfaces *cf.* the sphere:

surface	integral curvature	χ	$\delta\chi(sphere)$
sphere	4π	2	0
cross-surface	2π	1	-1
torus	0	0	-2

• *Euler's equation* relates # vertices (v), edges (e) and faces (f) making a tiling in a surface with χ:

$$v - e + f = \chi$$

Removing a disc ($f = -1$) from any closed surface therefore incurs a cost $\delta\chi = -1$.

- A handle ($handle$) is a torus minus one disc, so χ per $\circ$=-1

- A cross-cap ($\times$) is a cross-surface minus one disc, so χ per $\times$=0

So the manifold $\circ^i *^j \times^k$, built from a sphere with $i + j + k$ removed discs, i handles, and k cross-caps has characteristic:

$$\chi(\circ^i *^j \times^k) = 2-(i+j+k)+i(-1)+k(0) = 2-(2i+j+k)$$

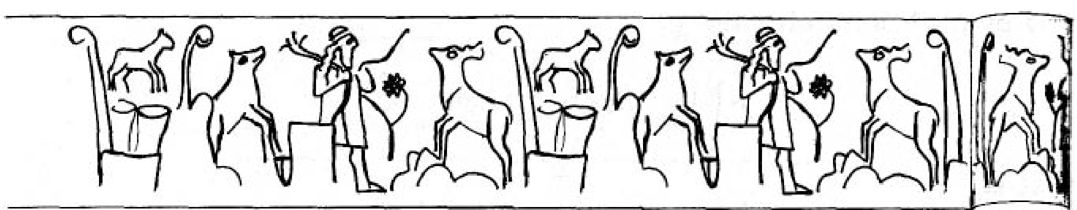
Evidently, most interesting manifolds have negative χ; they are hyperbolic!

The universal cover

Zipping up a 2-periodic planar pattern gives a cylinder, and a torus.

Unzipping an infinite patterned layering on the torus give a 1- or 2-periodic pattern in $\mathbf{E^2}$, depending on how many zip-pairs we unzip.

Equivalently, we print by rolling endlessly and tile the relevant *universal cover* ($\mathbf{S^2}$, $\mathbf{E^2}$, $\mathbf{H^2}$):

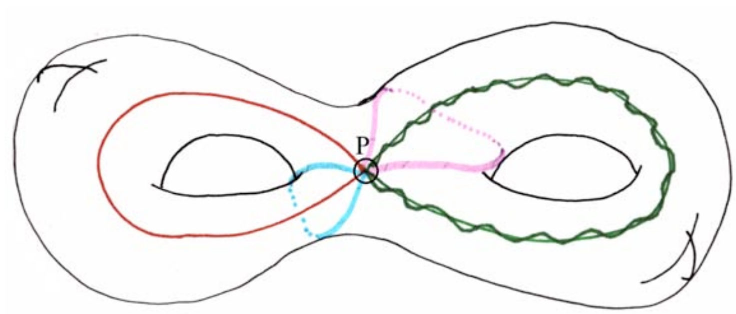


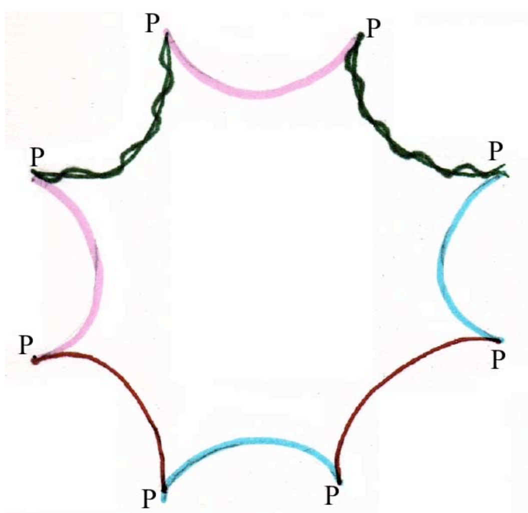
John Stillwell

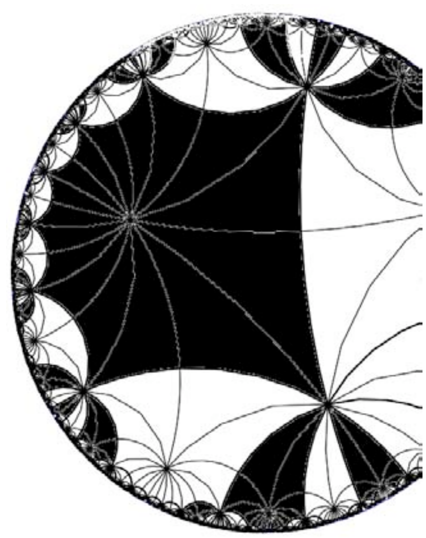
A canonical form for an unzipped manifold $\circ^n$ is a polygon with $4n$ sides, as $2n$ zips can be traced from a single point on the manifold, two along each handle.

Since each zip gives an independent lattice vector, printing the manifold $\circ^i$ gives a periodic pattern with up to $2i$ independent lattice vectors.

example 1: the genus-2 torus, ∘∘, has $\chi = -2$, so it prints onto $\mathbf{H^2}$.
It has 4 independent zip-pairs, allowing 4 translations in its universal cover:



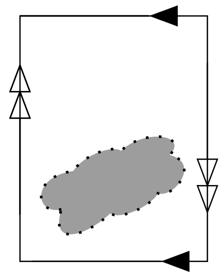




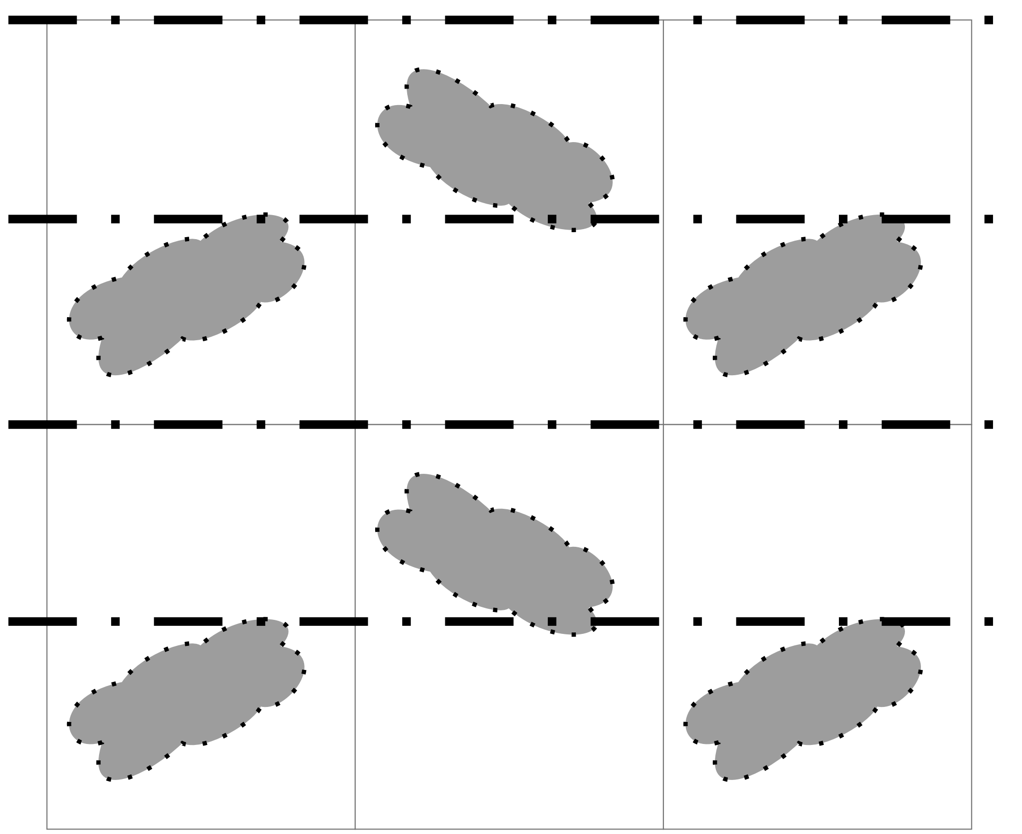
The illustrated pattern of zips gives a simply-connected octagon in $\mathbf{H^2}$ with maximum symmetry $*288$.
The universal cover has Schläfli symbol $\{8,8\}$

◦ (handles) induce translations

example 2: printing a patterned Klein bottle, $\times\times$
($\chi = 0$, so it prints to $\mathbf{E^2}$):



Zip-pair orientation for $\times\times$ defines pgg pattern

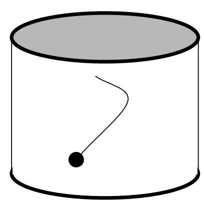


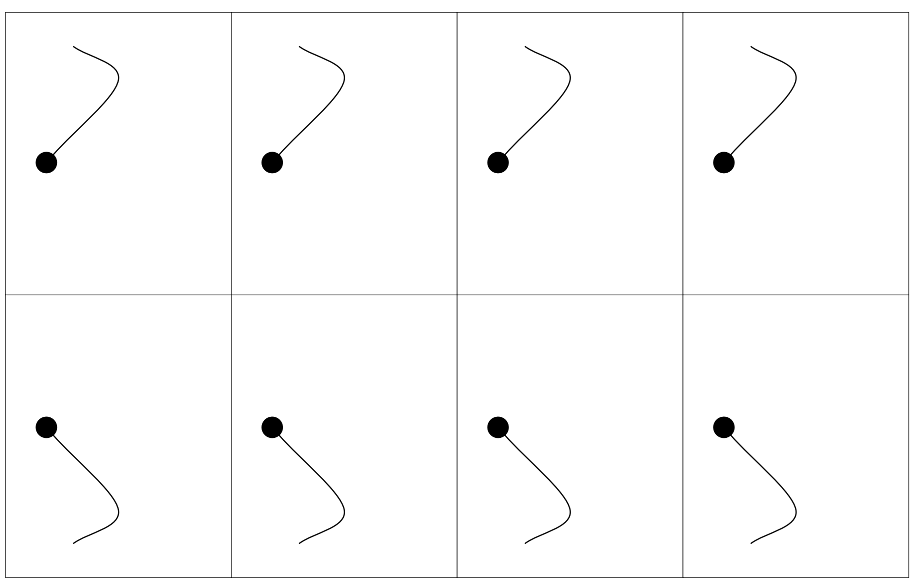
example 3: printing a bounded manifold, $**$
($\chi = 0$, so it prints to $\mathbf{E^2}$):

1. roll the manifold about its central axis, filling a single strip of $\mathbf{E^2}$

2. turn through 180° at each point on the $*$ boundaries

3. continue printing from the other side



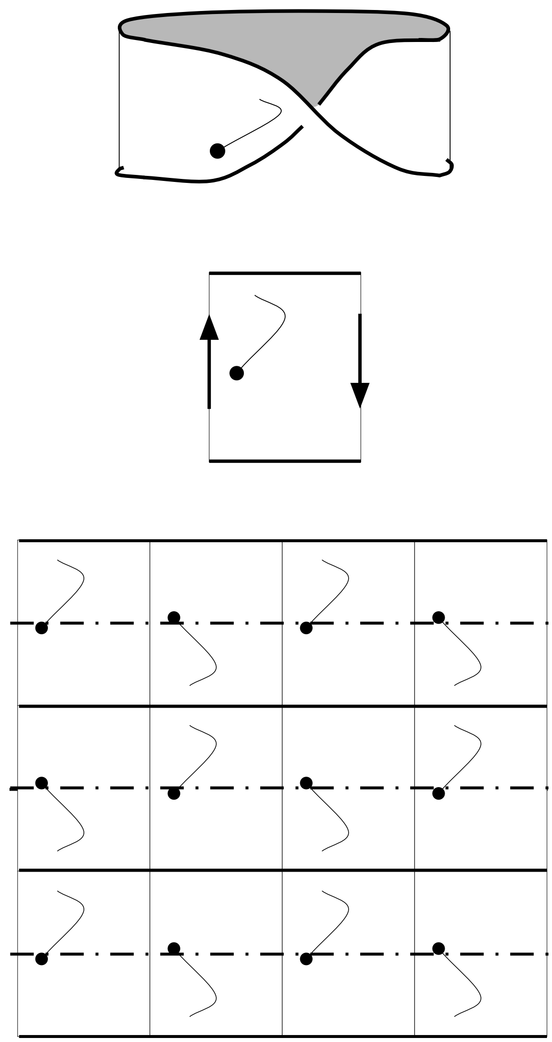


Zip-pair orientation for $**$ defines pmm pattern

$*$ boundaries are mirrors

example 4: printing a bounded non-oriented manifold, $*\times$ (Möbius strip)
($\chi = 0$, so it prints to $\mathbf{E^2}$):

1. unzip the manifold, giving a pair oppositely oriented sides
2. replicate along one axis according to edge orientations
3. replicate along other axis by reflecting in the boundary



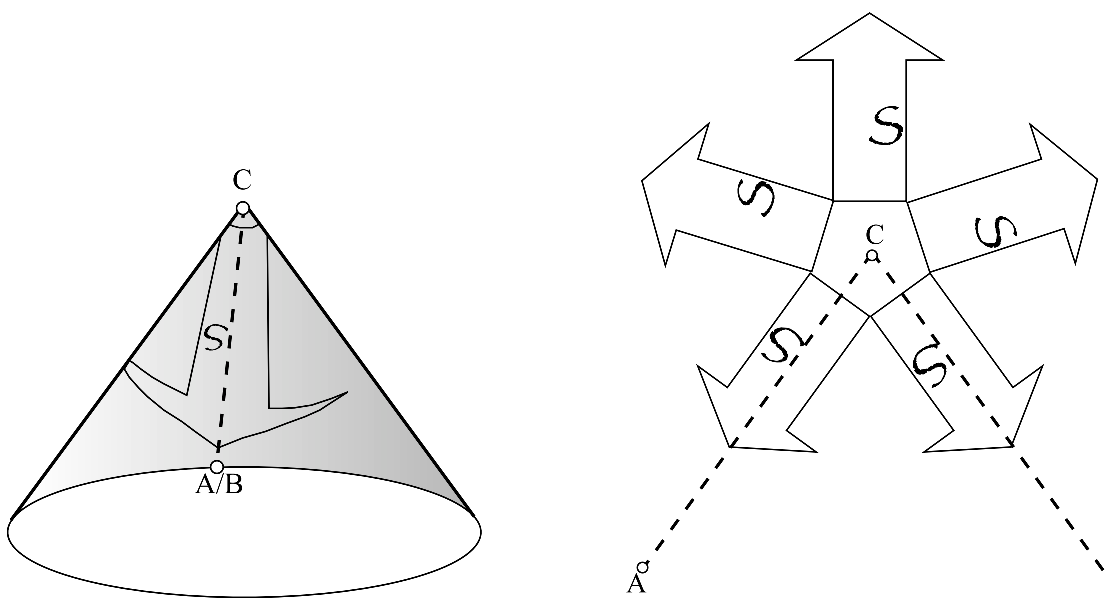
Zip-pair orientation for $*\times$ defines cm pattern

$\times$ (cross-caps) induce glides

From manifolds to orbifolds

If we add (non-manifold) **cone points**, we have a complete description of $2D$ orbifolds.

1. Cone points not coincident with mirrors:



χ due to a cone point of order A is $\frac{A-1}{A}$ (from integral curvature of a cone)

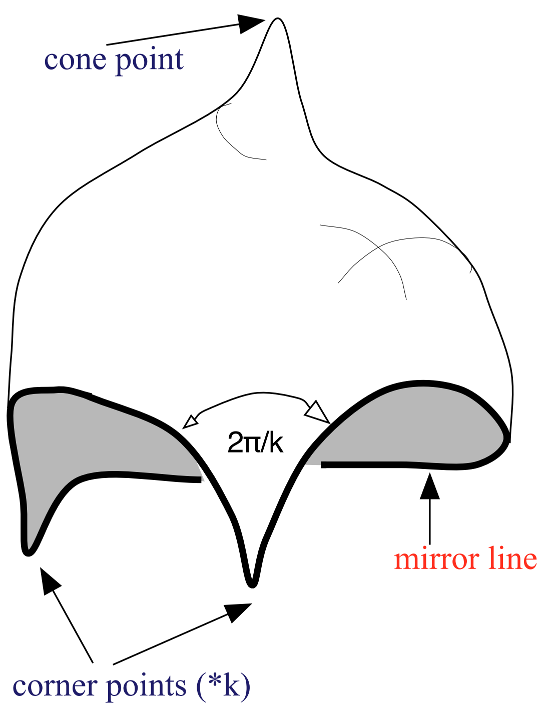
Cone-point of order A

induces A-fold rotational symmetry

corner point of order k $(*k)$

2. Cone point on k mirrors:

χ due to a corner point of order k is $\frac{k-1}{2k}$



χ of any orbifold

- Build an orbifold by zipping modules into the base orbifold: the sphere.

- The Euler-Poincaré characteristic, χ of the resulting orbifold is given by summing χ_{mod} over all modules:

Orbifold module	symbol	2D symmetry	χ_{mod}
Sphere	(-1)	identity	+2
Handle	$\circ$	translation	-2
Cone-point (order A)	A	A-fold centre of rotation	$-\frac{A-1}{A}$
Boundary	$*$	mirror line	-1
Corner point (angle $\frac{\pi}{m}$)	$(*)m$	intersection of m mirrors	$-\frac{m-1}{2m}$
Cross-cap	$\times$	glide line	-1

e.g. $\circ^h ABC * ijk * lm\times^c$

(conventionally written as $ABC * ijk * lm\times^{c+2h}$):

$$
\begin{aligned}
\chi \quad = \quad & 2 - \{2.h + \frac{A-1}{A} + \frac{B-1}{B} + \frac{C-1}{C} + \\
& 1 + \frac{i-1}{2i} + \frac{j-1}{2j} + \frac{k-1}{2k} \\
& +1 + \frac{l-1}{2l} + \frac{m-1}{2m} + 1.c\}
\end{aligned}
$$

• *ANY* character string combining integers $(1, 2, \ldots)$, handles $(\circ)$, mirrors $(*)$ and crosscaps $(\times)$ is a valid orbifold symbol

EXCEPT A and AB, $*A$ and $*AB$, where $A \neq B$.

(The notation system, zip concept and rules are due to John Conway).

Exercise 4 *Can you see why these four Conway symbols are bad orbifolds?*

SOLUTION to EXERCISE 4: Look at forming these as symmetries on the sphere. The presence of a rotational symmetry centre (on or off a mirror) induces the same order centre at the antipode on the sphere. Thus, a pair of cone of corner points is needed.

There is some redundancy in words:

- Cone points and distinct mirror boundaries (with accompanying corner points) can be shuffled at will

- corner points belonging to a single mirror boundary can be cyclically reordered.

- ordering of corner points belonging to any mirror string can be reversed for non-orientable manifolds; all corner points must be simultaneously reversed for orientable manifolds

e.g.

$$\circ 246 * 34 * 456 = \circ 462 * 654 * 43$$
$$246 * 34 * 456 \times = 426 * 654 * 34 \times$$

Exercise 5 *Determine the characteristic and universal cover of the following orbifolds (preferably without looking at the Tables below)!:*

1. $*632$

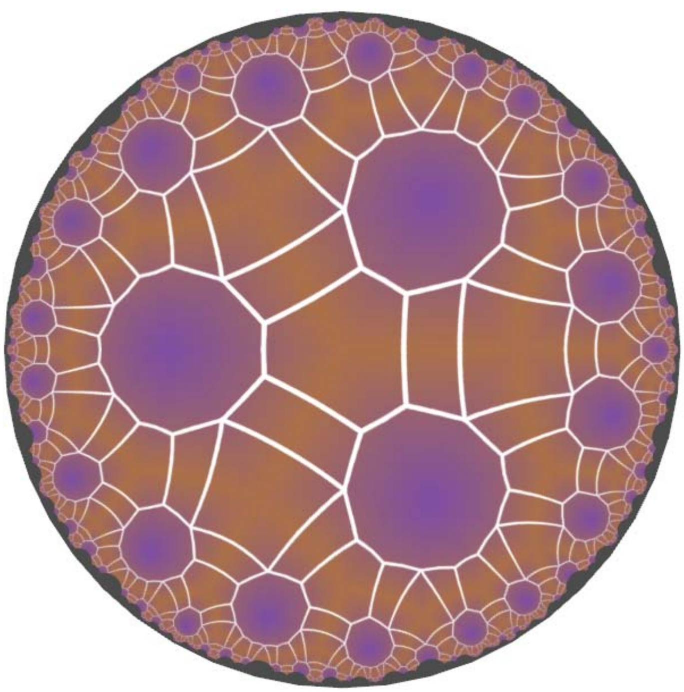
2. 632

3. $*532$

4. $*732$

5. $\circ^3$

6. *Determine the orbifold type for the pattern below:*



SOLUTIONS to EXERCISES 5

1. 0, $\mathbf{E^2}$

2. 0, $\mathbf{E^2}$

3. $\frac{1}{60}$, $\mathbf{S^2}$

4. $\frac{-1}{84}$, $\mathbf{H^2}$

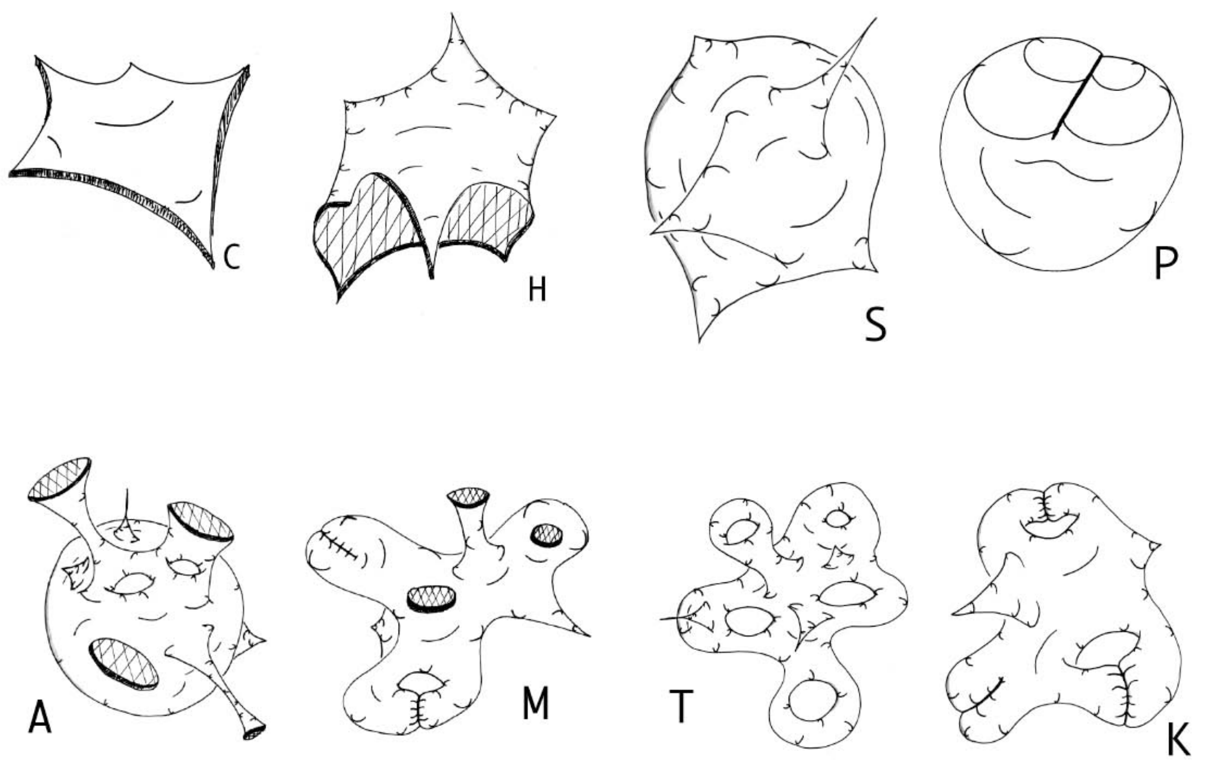
5. -4, $\mathbf{H^2}$

6. $*344$

Orbifold Taxonomy

- We propose the following families:

Coxeter, Hat, Stellate, Projective, Annulus, Möbius, Torus and Klein



The two rows collect simply- and multiply-connected orbifolds.

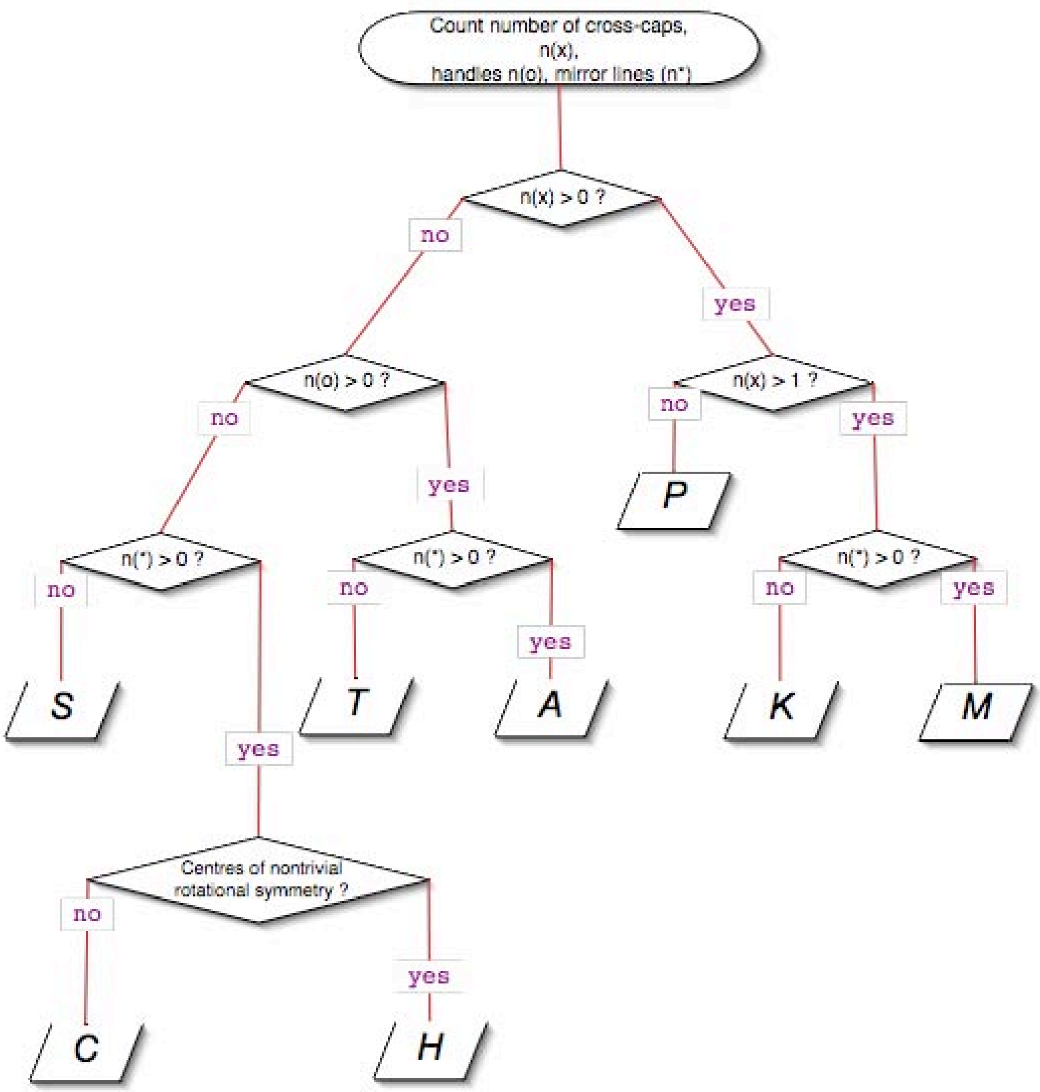
An *aide memoire*:

Churlish Hatred of Symbols Prevents Acquisition of Modern Topological Knowledge.

The relevant family can be read from the Conway orbifold symbol at a glance . . .

	Abbreviation	Symbol
Simply connected:		
Mirror edges	C	$*ij\ldots k$
	H	$A\ldots B*ij\ldots k$
Edge-free	S	$A\ldots B$
	P	$A\ldots B\times$
Multiply connected:		
Mirror edges	A	$\circ\cdots\circ A\ldots B*i\ldots j\cdots*k\ldots l$
	M	$A\ldots B*i\ldots j*k\ldots l*\cdots\times\cdots\times$
Edge-free	T	$\circ\cdots\circ A\ldots B$
	K	$A\ldots B\times\cdots\times$

A decision tree to determine orbifold families:



Why bother with orbifolds?

1. Conway symbols allow for coherent notation across all $2D$ geometries.

2. Symbols unify topology with symmetry

3. Systematic enumeration of $2d$ tilings and nets is possible, using Delaney-Dress theory

4. Almost trivial enumeration of point groups (on $\mathbf{S^2}$) and plane groups (on $\mathbf{S^2}$):

$$\text{point groups iff } \chi > 0$$

$$\text{plane groups iff } \chi = 0$$

Orbifolds on $\mathbf{E}^2$ and $\mathbf{S}^2$: Plane and Point Groups

Family	**Conway symbol**	
Coxeter	$*632$	$p6m$
(C)	$*442$	$p4m$
	$*333$	$p3m1$
	$*2222$	pmm
	$**$	pm
Hat	$4*2$	p4g
(H)	$3*3$	p31m
	$2*22$	cmm
	$22*$	pmg
Stellate	632	p6
(S)	442	p4
	333	p3
	2222	p2
Projective plane (P)	$22\times$	pgg
Annulus (A)	$**$	pm
Möbius band (M)	$*\times$	cm
Torus (T)	$\circ$	p1
Klein bottle (K)	$\times\times$	pg

Family	Order	Characteristic	Orbifold		
Coxeter (C)	120	$\frac{1}{60}$	$*235$	–	I_h
	48	$\frac{1}{24}$	$*234$	$m\overline{3}m$	O_h
	24	$\frac{1}{12}$	$*233$	$\overline{4}3m$	T_d
	24	$\frac{1}{12}$	$*226$	$6/mmm$	D_{6h}
	16	$\frac{1}{8}$	$*224$	$4/mmm$	D_{4h}
	12	$\frac{1}{6}$	$*223$	$\overline{6}2m$	D_{3h}
	8	$\frac{1}{4}$	$*222$	mmm	D_{2h}
	4k	$\frac{1}{2k}$	**22k*	–	D_{kh}
	12	$\frac{1}{6}$	$*66$	$6mm$	C_{6v}
	8	$\frac{1}{4}$	$*44$	$4mm$	C_{4v}
	6	$\frac{1}{3}$	$*33$	$3m$	C_{3v}
	4	$\frac{1}{2}$	$*22$	$mm2$	C_{2v}
	2	1	$*$	m	C_s
	2k	$\frac{1}{k}$	$*kk$	–	C_{kv}
Hat (H)	24	$\frac{1}{12}$	$3*2$	$m\overline{3}$	T_h
	12	$\frac{1}{6}$	$2*3$	$\overline{3}m$	D_{3d}
	8	$\frac{1}{4}$	$2*2$	$\overline{4}2m$	D_{2d}
	12	$\frac{1}{6}$	$6*$	$6/m$	C_{6h}
	8	$\frac{1}{4}$	$4*$	$4/m$	C_{4h}
	6	$\frac{1}{3}$	$3*$	$\overline{6}$	C_{3h}
	4	$\frac{1}{2}$	$2*$	$2/m$	C_{2h}
	2k	$\frac{1}{k}$	$k*$	–	C_{kh}
Stellate (S)	60	$\frac{1}{30}$	235	–	I
	24	$\frac{1}{12}$	234	432	O
	12	$\frac{1}{6}$	233	23	T
	12	$\frac{1}{6}$	226	622	D_6
	6	$\frac{1}{3}$	66	6	C_6
	6	$\frac{1}{3}$	223	32	D_3
	3	$\frac{2}{3}$	33	3	C_3
	8	$\frac{1}{4}$	224	422	D_4
	4	$\frac{1}{2}$	44	4	C_4
	4	$\frac{1}{2}$	222	222	D_2
	2	1	22	2	C_2
	k	$\frac{2}{k}$	kk	–	C_k
Projective (P)	6	$\frac{1}{3}$	$3\times$	$\overline{3}$	C_{3i}
	4	$\frac{1}{2}$	$2\times$	$\overline{4}$	S_4
	2	1	$\times$	$\overline{1}$	C_i

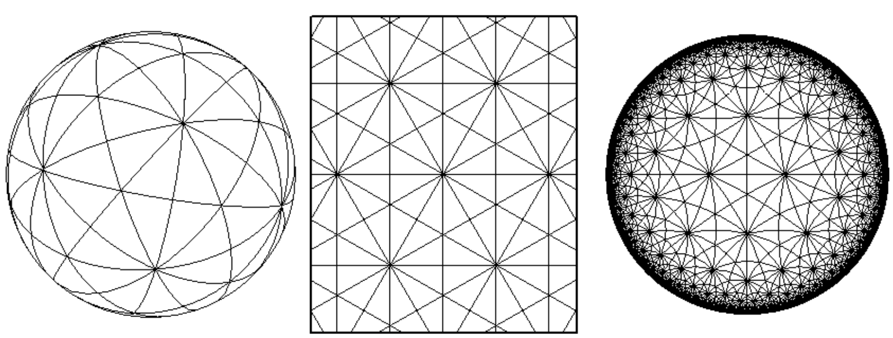
Symmetry mutation within an orbifold family:

Orbifold Species

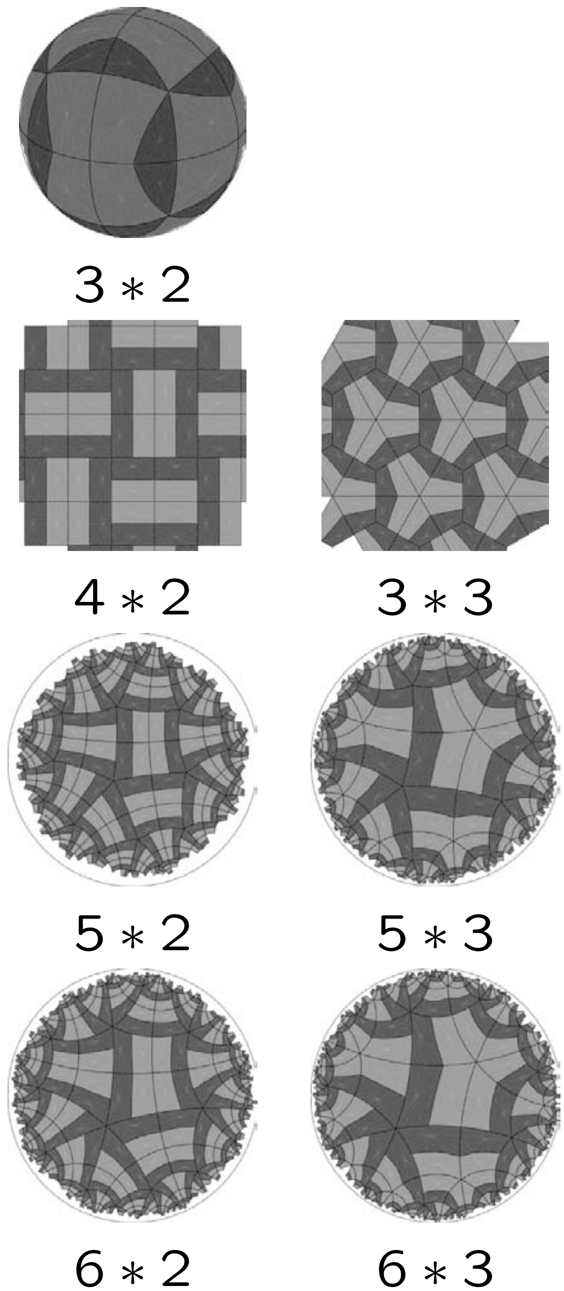
Changing the values of the integers - **mutates** the orbifold within a single species.

Allows single families to span all three 2D geometries, from $\mathbf{S^2}$ to $\mathbf{H^2}$.

e.g. *Coxeter* orbifold $*235 \rightarrow *236 \rightarrow *237$



$\mathbf{S^2} \rightarrow \mathbf{E^2} \rightarrow \mathbf{H^2}$

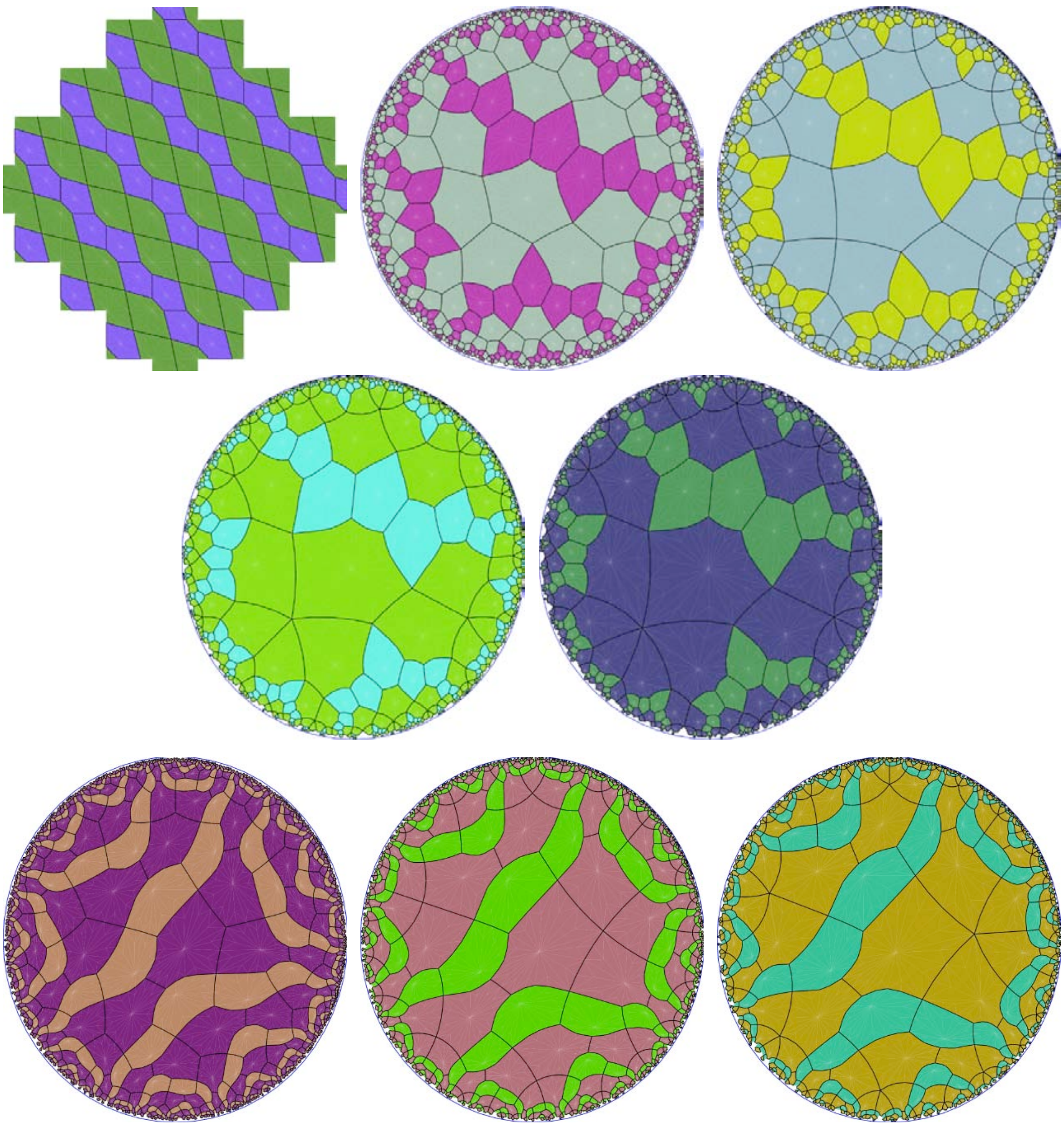


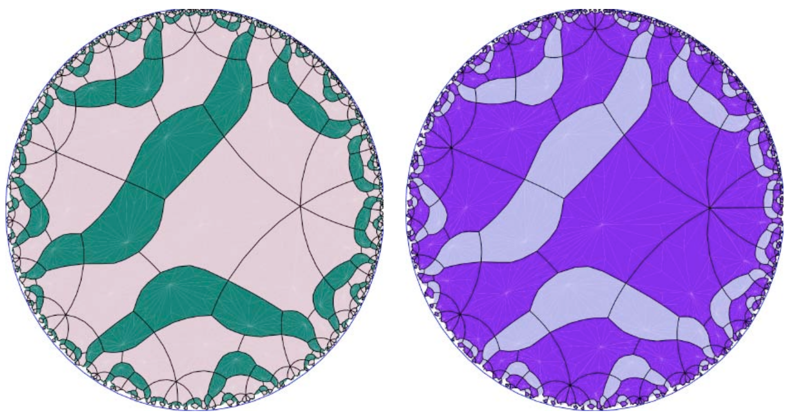
Hat mutations:
All examples have Conway symbol $k * m$, where k varies between 3 and 6 and m between 2 and 3.

Exercise 6 *1. Identify the orbifolds in the series of patterns in the Figures and draw their topological form. (The first images of each series are pentagonal tilings of the flat plane, discovered by Marjorie Rice.)*

2. How may species are collected here?

3. Sketch a possible shape for these orbifolds (one for each species).



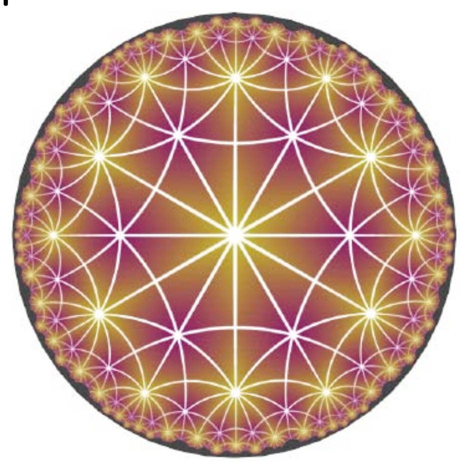


SOLUTIONS to EXERCISES 6:

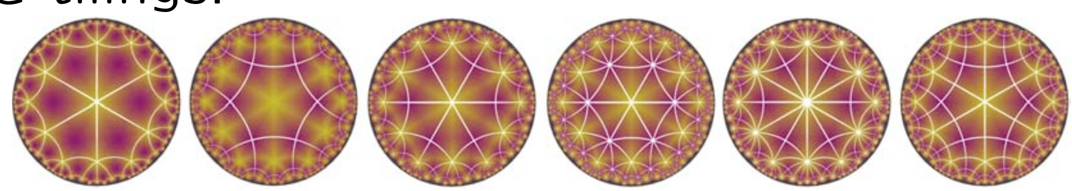
1. 2222,3222,4222,5222,6222,32×,42×,52×,62×,72×.
2. 2.
3.

Decorating orbifolds to get hyperbolic nets

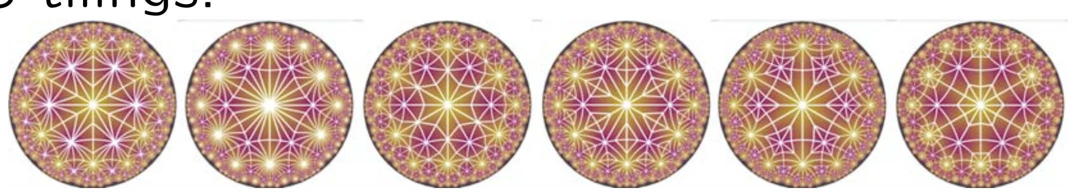
- Use Delaney-Dress tiling theory to enumerate all tilings up to equivariance
  – i.e. tilings with distinct 2D topology and 2D symmetry
- Done this for Fundamental tilings, 131 hyperbolic groups:



- FG tilings:



- FS tilings:



- FSG- and FSGG tilings.

Group-subgroup relations and orbifolds

$|\chi| \sim$ area of asymmetric domain in universal cover (by global Gauss-Bonnet)

- Any point group $\mathcal{P}$ acts on the sphere, $\mathbf{S^2}$ ($\chi = 2$), so:

$$\mathcal{O}(\mathcal{P}) = \frac{2}{\chi(\mathcal{P})}$$

- Hyperbolic groups may act on a multiply-connected hyperbolic manifold (whose topology is given by χ) in $\mathbf{H^2}$.
  The order of such a group, $\mathcal{H}$, in that manifold is:

$$\mathcal{O}(\mathcal{H}) = \frac{\chi}{\chi(\mathcal{H})}$$

- Highest order point group has smallest positive χ: $*235$

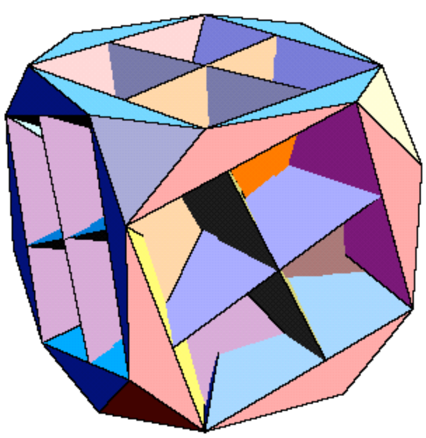
$$\chi = \frac{1}{60}$$

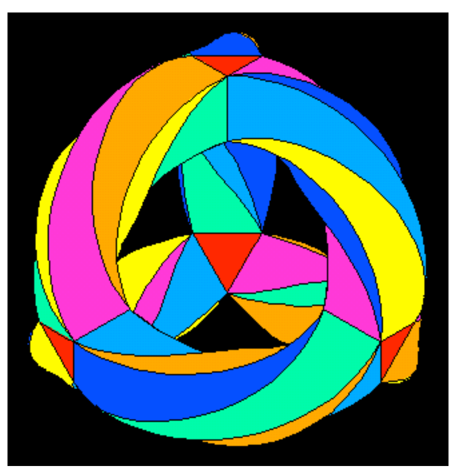
- Highest order hyperbolic symmetry group has largest negative χ: $*237$

$$\chi = -\frac{1}{84}$$

This cannot be realised in 3-space without distortion . . .

Most symmetric embeddings are on the genus 3 torus ($*237/\circ^3$)





`www.math.uni-siegen.de/wills/klein/;`
`gregegan.customer.netspace.net.au/SCIENCE/KleinQuartic/KleinQuartic.html`

Highest order hyperbolic symmetries:

Cost	Class	orbifolds
$\frac{-1}{84}$	C	$*237$
$\frac{-1}{48}$	C	$*238$
$\frac{-1}{42}$	S	237
$\frac{-1}{40}$	C	$*245$
$\frac{-1}{36}$	C	$*239$
$\frac{-1}{30}$	C	$*23(10)$
$\frac{-5}{132}$	C	$*23(11)$
$\frac{-1}{24}$	C S	$*246,\ *23(12),\ *334$ 238
$\frac{-7}{156}$	C	$*23(13)$
$\frac{-1}{21}$	C	$*23(14)$
$\frac{-1}{20}$	C S	$*23(15)$ 245

Can we realise any of these in $\mathbf{E^3}$?

e.g. $*\mathbf{246}$:

Embedding of the hyperbolic manifold $*246/\circ\circ\circ$ (the genus 3 torus with $*246$ internal symmetry), as a branched (double) covering of the sphere, symmetry $*234$.

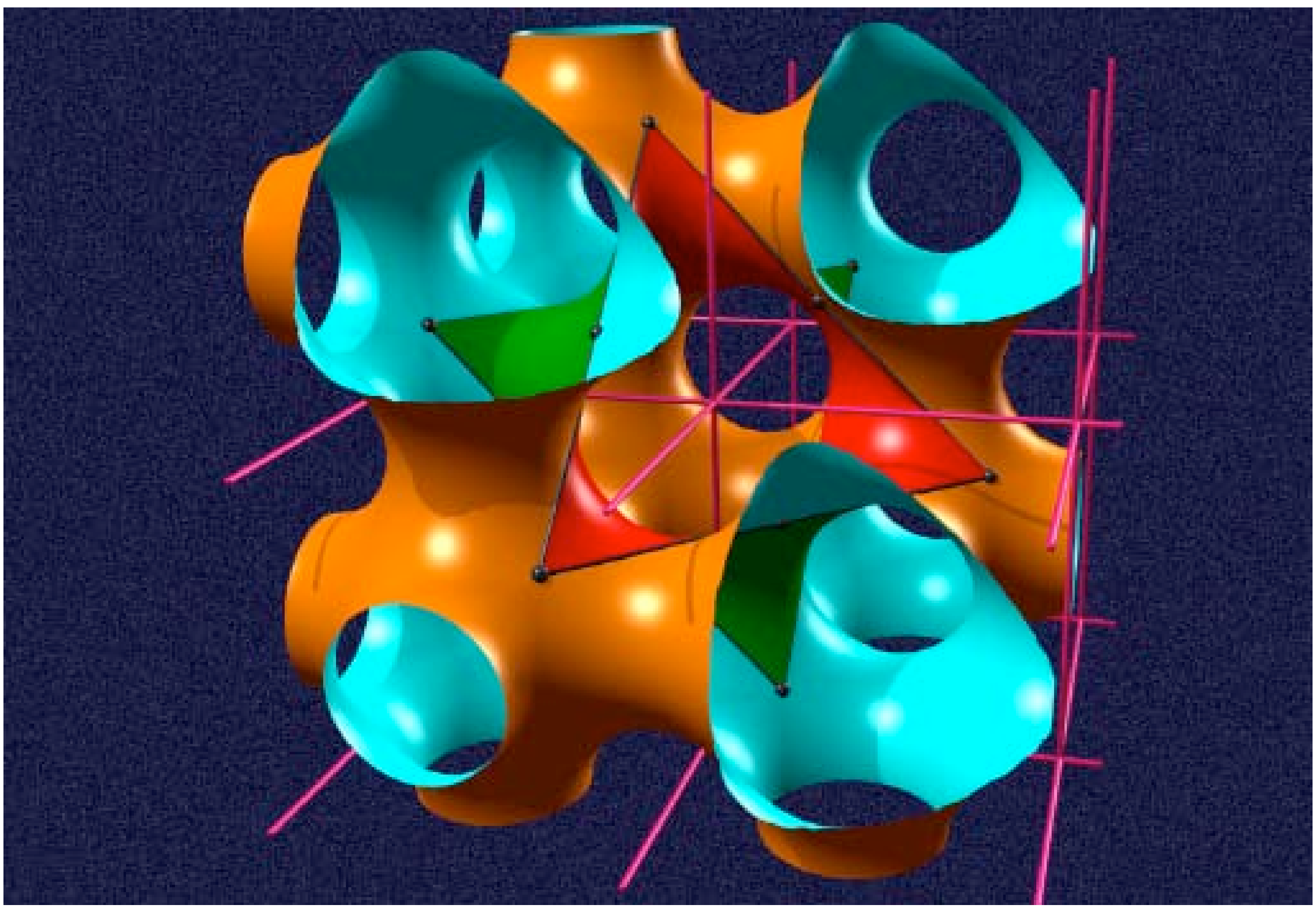
Exercise 7 *Alternative embeddings of* $*246/\circ\circ\circ$.

Find the orbifold symbols and sub-group orders of the elliptic (sub-)groups (point groups) of the three embeddings in $\mathbb{E}^3$ *of tilings by* $*246$ *triangles shown below.*

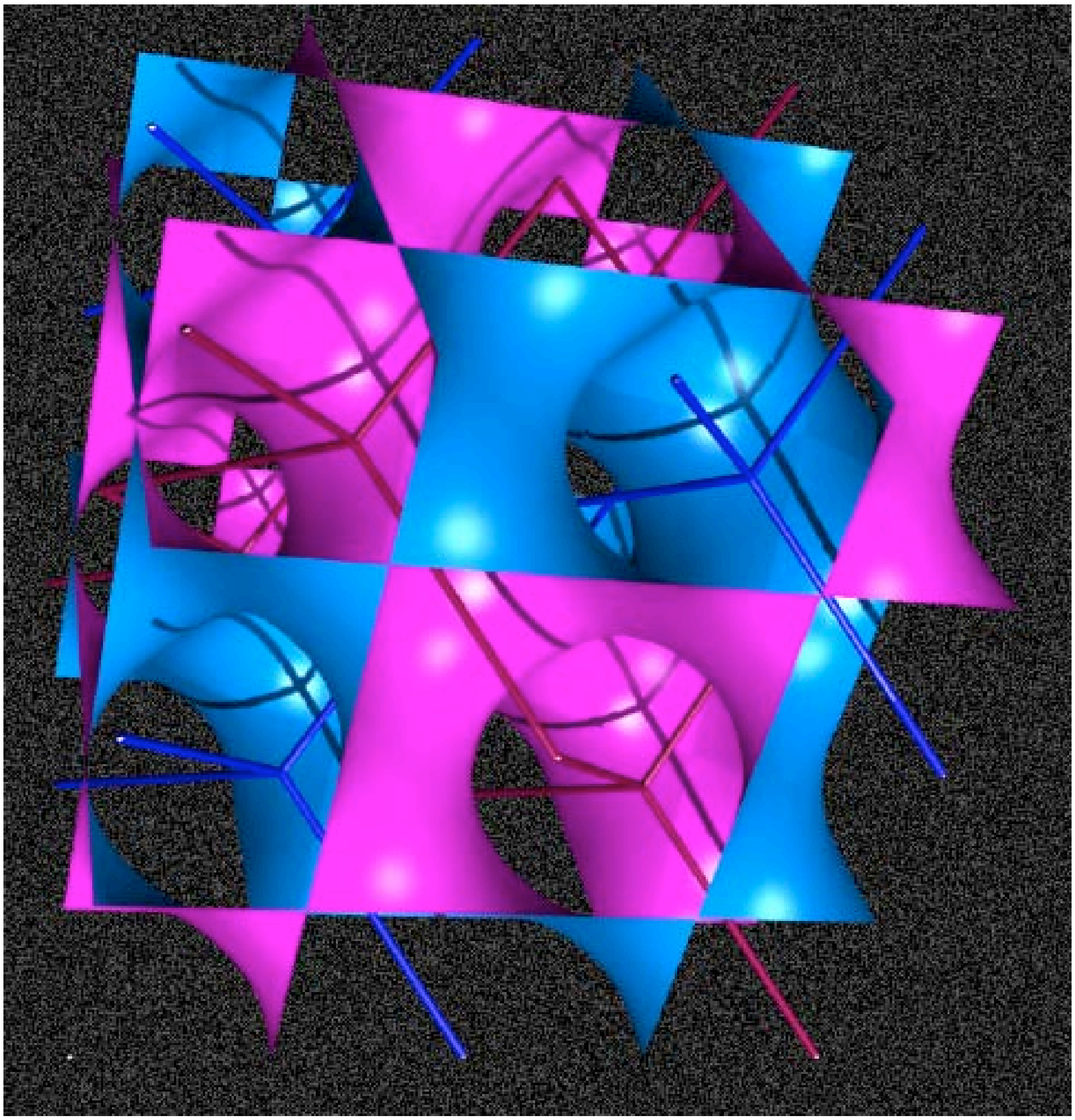
SOLUTIONS to EXERCISES 7: Clockwise from top left: $3\times, 2*2, *224$

$*246$ can be faithfully embedded in $\mathbf{E^3}$ iff the embedding surface has unbounded genus:

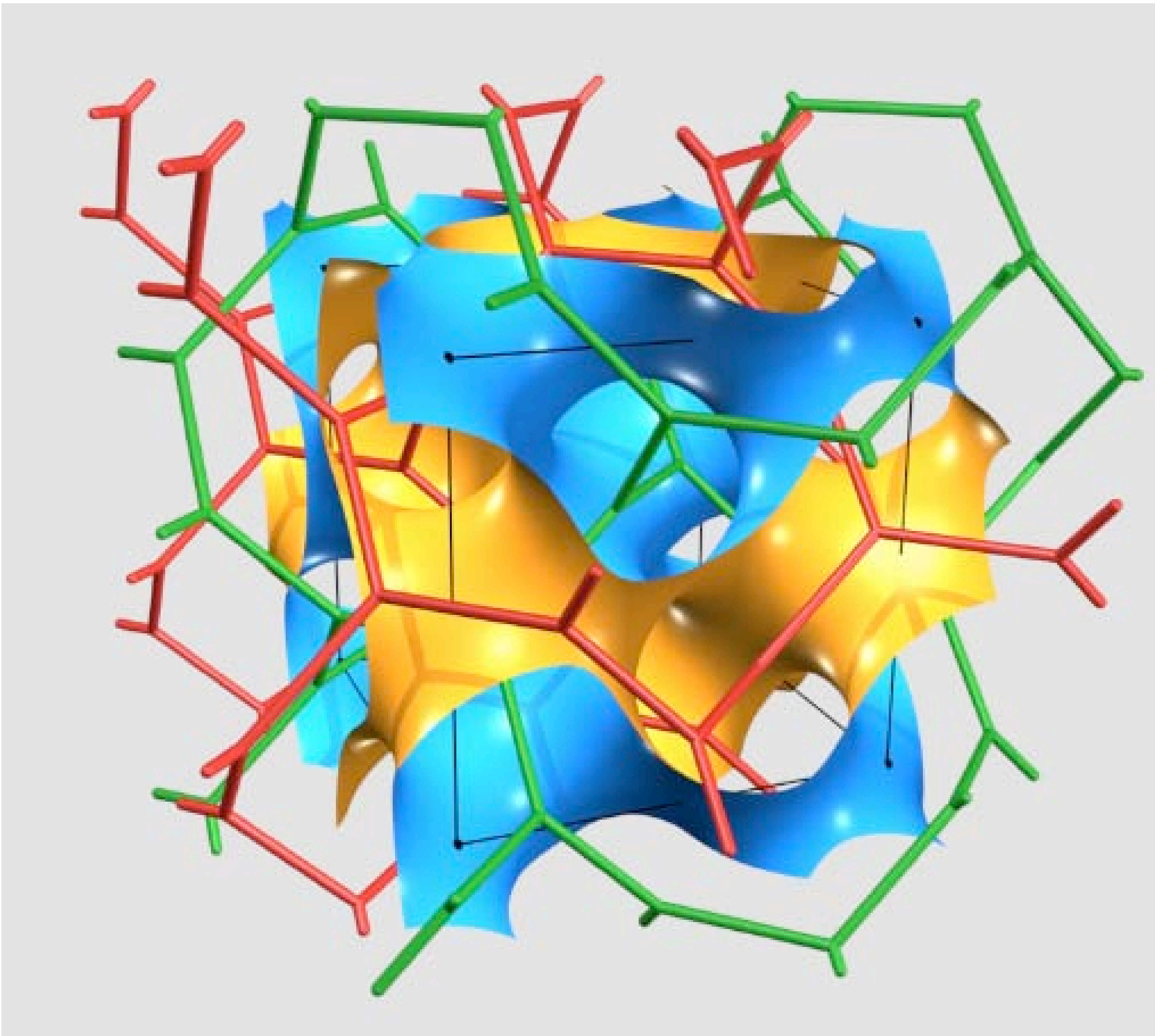
Simplest geometrical realisations are the 3-periodic minimal surfaces:



P surface

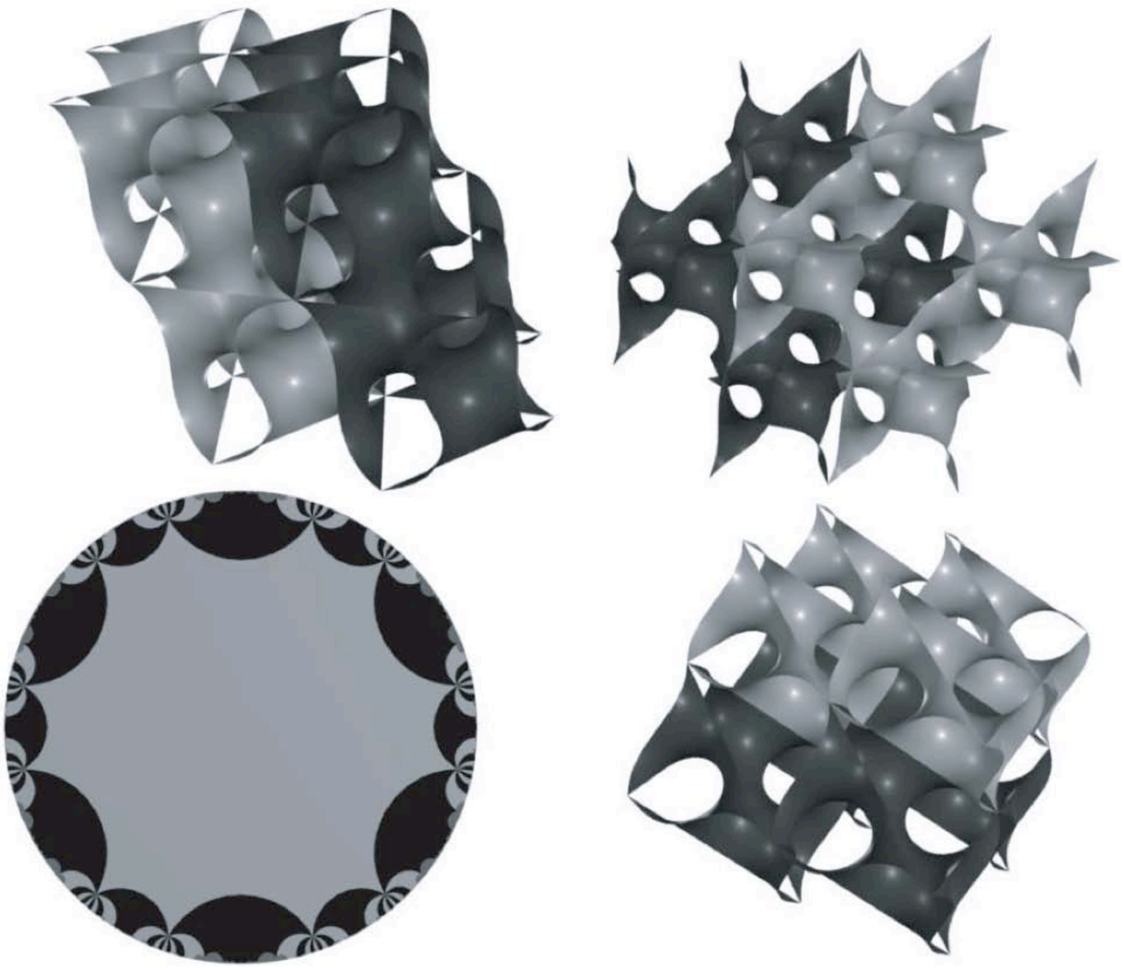


$D(iamond)$ surface



G surface

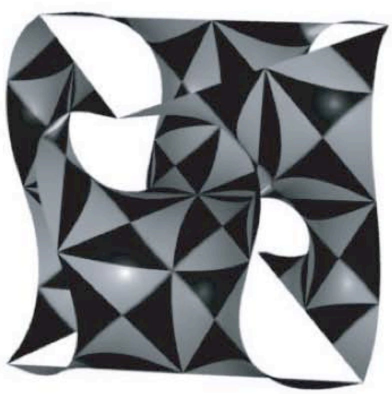
• They have identical universal covers, since they have identical Gauss maps.

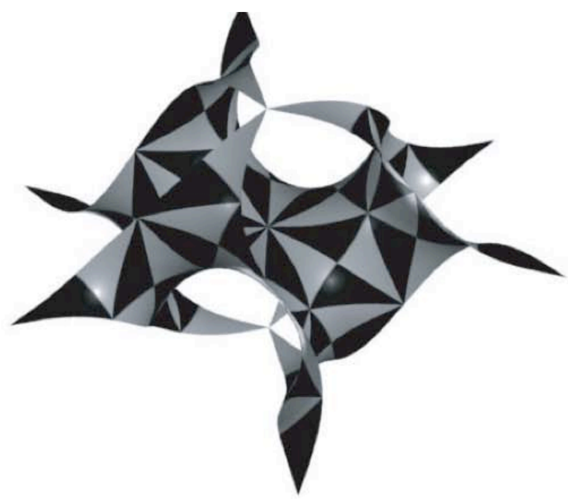


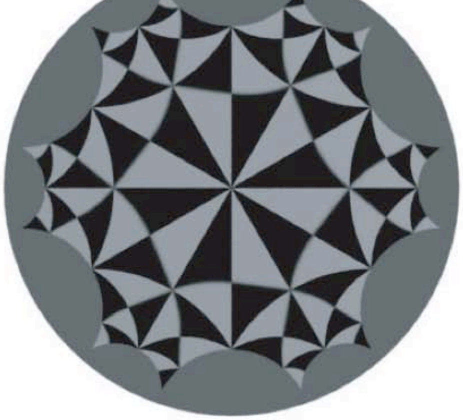
Any pattern in $\mathbb{H}^2$ whose orbifold symmetry is a sub-group of $*246$ can be faithfully embedded into $\mathbb{E}^3$ via these surfaces.

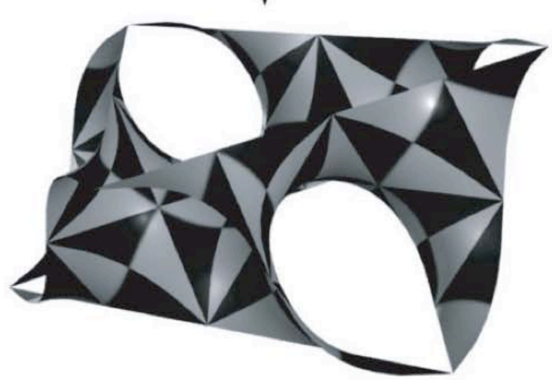
We choose patterns that respect all translational symmetries of these (cubic) surfaces.

- All three surfaces have euclidean unit cells (b/w)$\rightarrow$ $\chi = -4$, genus 3, (orientable) manifold $\circ^3$.

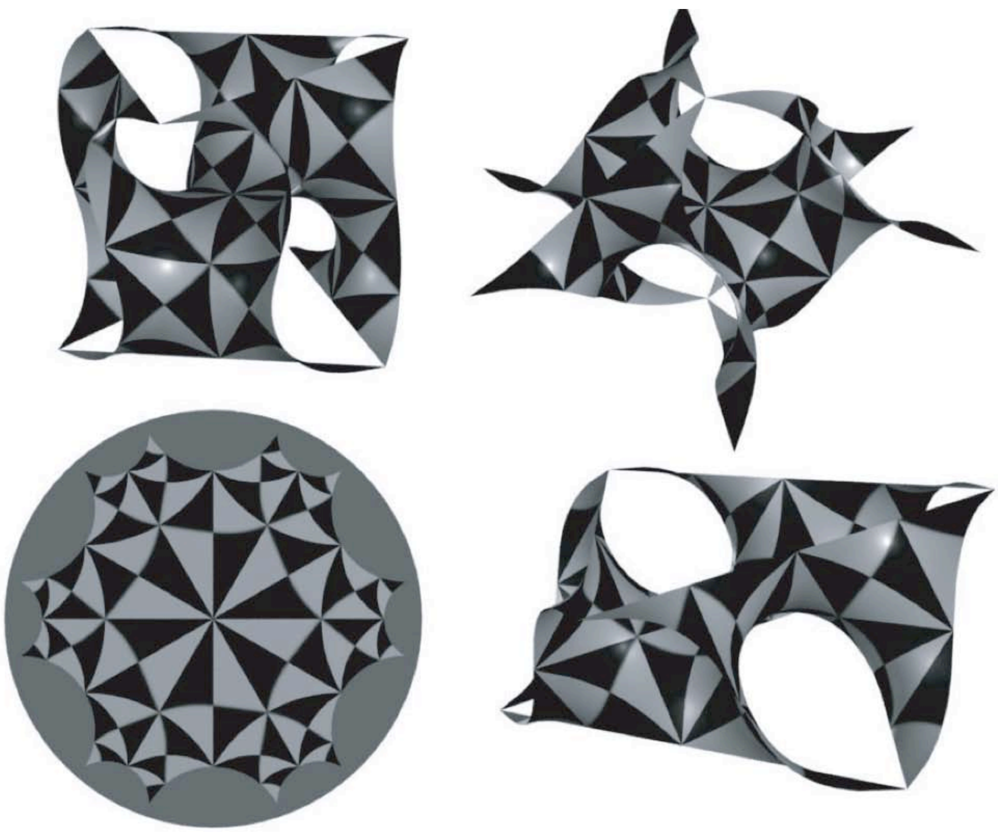








• Their pure translational hyperbolic sub-group $\circ^3$ map to a dodecagon in the universal cover.
• Euclidean translations are a subset of the 6 hyperbolic translations, $\{t_i, \tau_i\}, i = 1, 2, 3$, joining opposite long and short edges

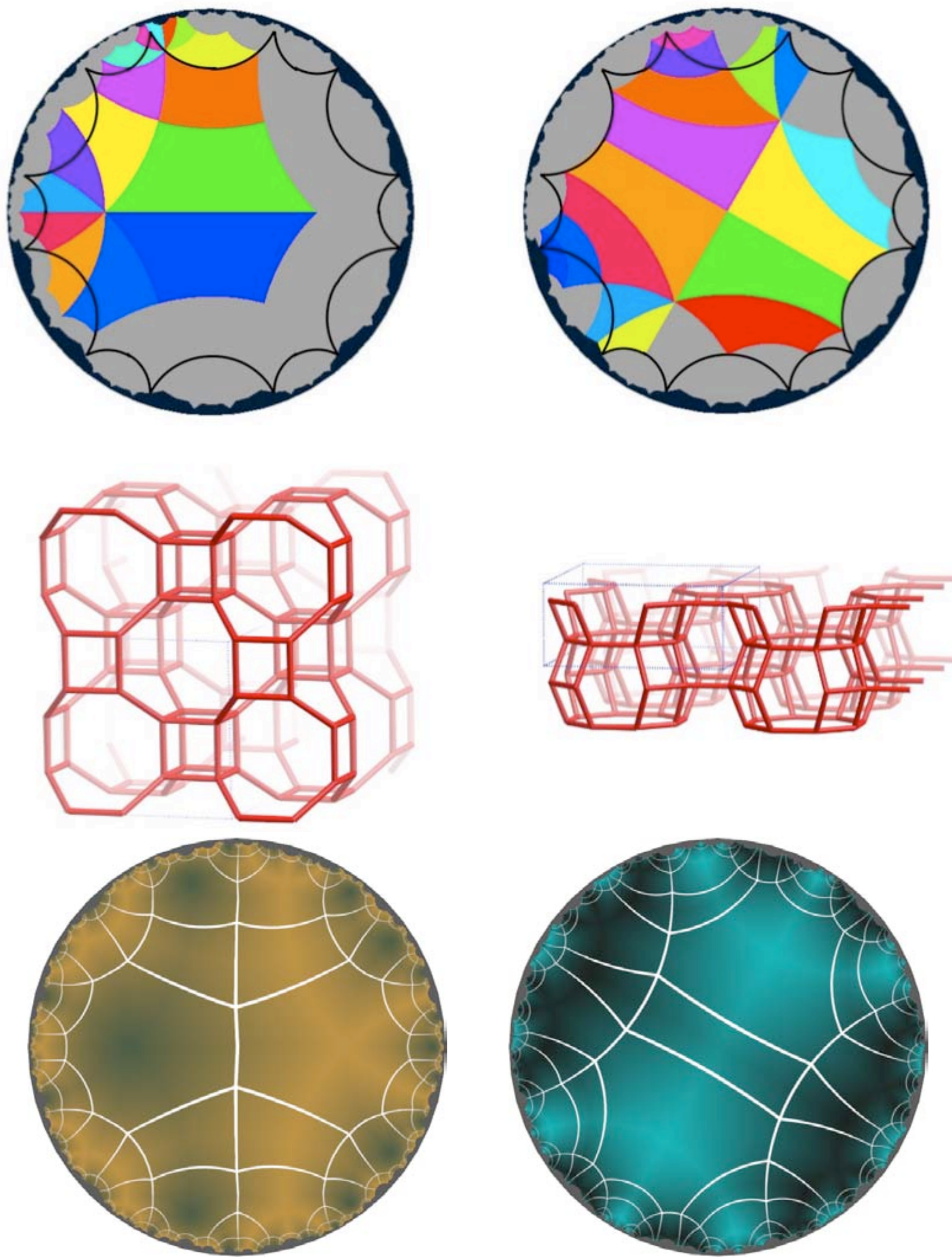


• P, G, D have different combinations of $\{t_i, \tau_i\} \rightarrow$ lattice vectors in $\mathbf{E^3}$.

Escher movie

Finding commensurate hyperbolic groups

- encode the finite group $*246/\circ^3$ by adding extra relations to the infinite group $*246$.
- these relations define the $\{t_i, \tau_i\}$ zip-pairs
- determine all groups that are super-groups of $\circ^3$ and sub-groups of $*246$
  - 131 *crystallographic hyperbolic* groups are found, belonging to 90 hyperbolic orbifolds.
  - 14 pure reflection groups, giving 11 distinct $Coxeter$ orbifolds (+3 isomorphisms)



isomorphisms of $*2244$: zeolite frameworks ACO and ATN

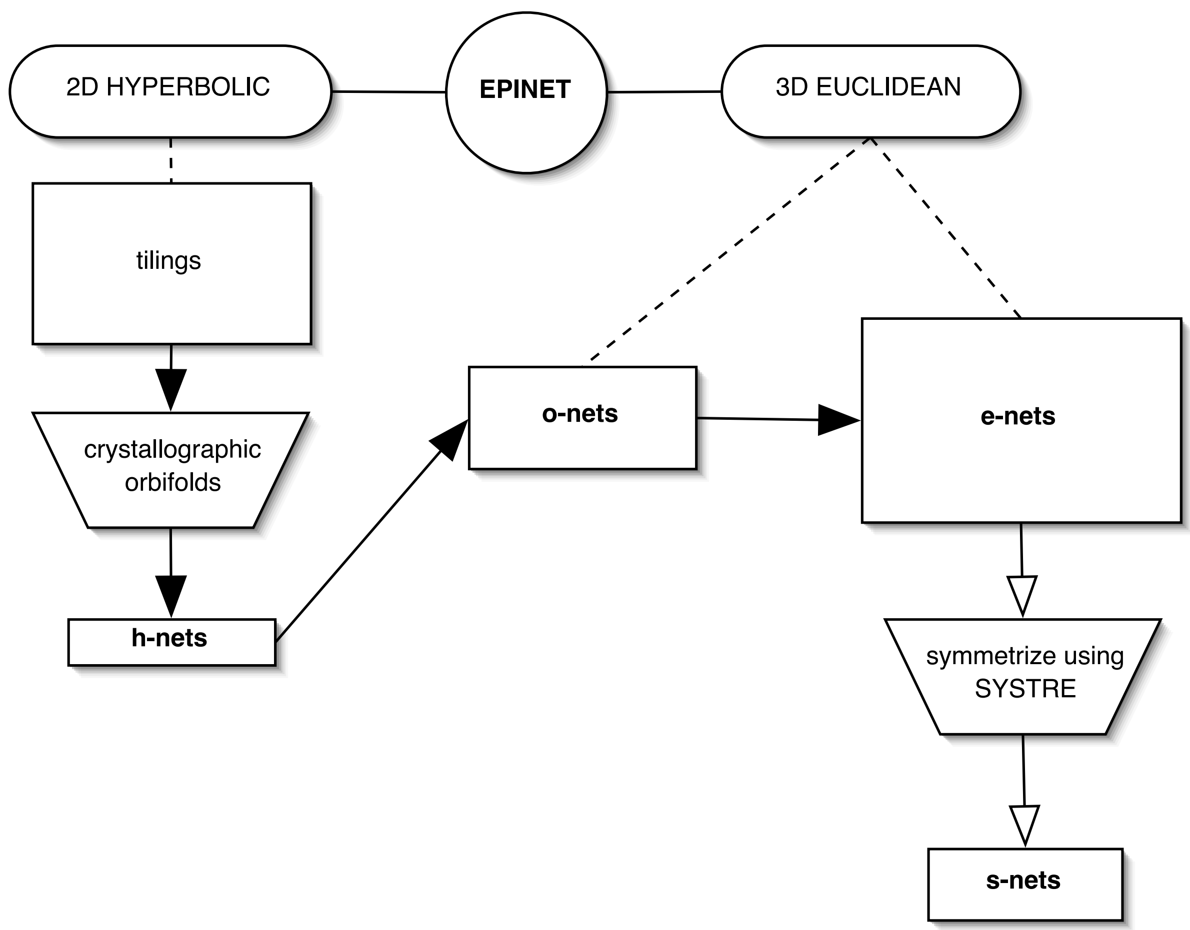
Crystallographic hyperbolic orbifolds

Family	Orbifolds
C	$*246$, $*266$, $*2224$, $*2223$, $*344$, $*2323$, $*2244$, $*22222$ ($*2^5$), $*2626$, $*2^6$, $*4^4$
H	$2*32$, $4*3$, $6*2$, $2*26$, $2*33$, $3*22$, $22*2$, $24*$, $2*44$, $4*22$, $2*222$, $22*3$, $2*2^4$, $22*22$, $44*$, $222*$, $22*2^4$, 2^4*
S	246, 266, 2223, 344, 2224, 2226, 2323, 2244, 2^4, 2^5, 2266, $2^4 3$, 2^6, 4^4, 2^8
P	$23\times$, $62\times$, $44\times$, $222\times$, $2^4\times$
A	$**2$, $*22*$, $*2*2$, $2**$, $**22$, $*3*3$, $*22*22$, $***$, $22**$, $\circ*$, $\circ**$
M	$*3\times$, $2*\times$, $*22\times$, $22*\times$, $**\times$, $*\times\times$ $*2^4\times$, $**\times\times$
T	$\circ 2$, $\circ 3$, $\circ 22$, $\circ 33$, $\circ\circ$, $\circ 2^4$, $\circ\circ\circ$
K	$2\times\times$, $3\times\times$, $22\times\times$, $\times^4$

EPINET = Euclidean Patterns in Non-Euclidean Tilings

`epinet.anu.edu.au`

- catalog of hyperbolic $h-nets$
- each $h-net \rightarrow e(pi)-net$
  – the geometry induced by the P, D, G reticulation
  ($e-nets$ can be knotted)
- $e-net \rightarrow s(ystre)-net$
  – the canonical barycentric placement of Delgado Friedrichs and O'Keeffe


2D HYPERBOLIC
EPINET
3D EUCLIDEAN
tilings
crystallographic orbifolds
h-nets
o-nets
e-nets
symmetrize using SYSTRE
s-nets

Using H^2 to get euclidean structures: s-nets

- build the $h-net$ from standard DD tiling theory applied to relevant hyperbolic orbifold ($\mathcal{O}$).

- the topology of the euclidean wrapping ($s-net$) can be determined directly from the $h-net$ in $\mathbf{H^2}$:

  –label all 96 $*246$ triangles in the dodecagon ($= \frac{\chi(\circ^3)}{\chi(*246)}$).

  –place symmetrically distinct vertices in a single domain of the orbifold $\mathcal{O}$ labelled domain

  – find the $*246$ triangles of all neighbouring vertices (at other ends of edges from distinct vertices)

  – determine the label of these triangles as a word in the $*246$ group

  –rewrite as t, τ word

  –rewrite t, τ word as a euclidean lattice vector

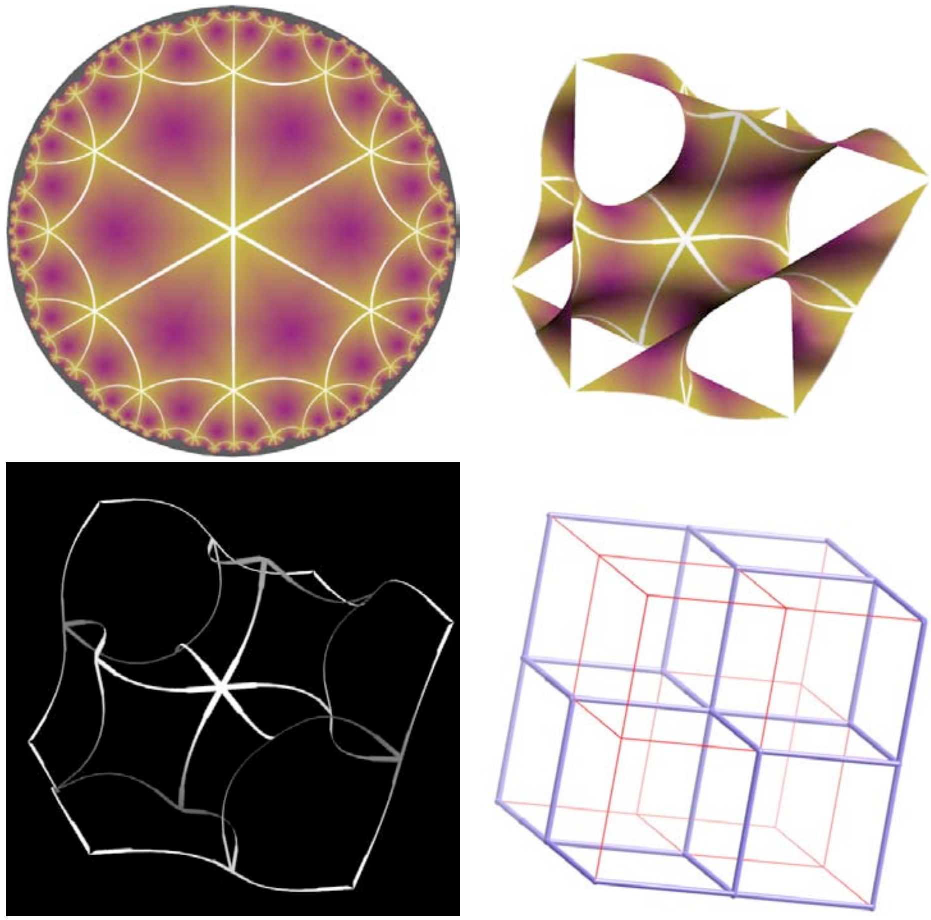
- this gives the *Systre key*, that identifies the *s-net* in $3D$.

Using H^2 to get euclidean structures: e-nets

We are working on a different scheme to get $e-nets$, that relies on the mapping of pure rotations (cone points) and translations in $\mathbf{H^2}$ to identical symmetry operators in $\mathbf{E^3}$ . . .

Suppose the $e-net$ has orbifold $\mathcal{O}$, on a TPMS whose Gauss map has (Coxeter) orbifold $\mathcal{G}$, where $\mathcal{G}$ is a symmetry mutation of $\mathcal{O}$:

- form sub-orbifold $\mathcal{O}'$ of $\mathcal{O}$, with no $*$ or $\times$ characters (by successive zipping of copies of $\mathcal{O}$, giving cones from corners and translations from cross-caps).
- find the corresponding sub-group of the Gauss map, $\mathcal{G}'$
- use a quasi-conformal map to transform all points in a single $\mathcal{O}'$ domain in $\mathbf{H^2}$ onto $\mathcal{G}'$, in $\mathbf{S^2}$ (á la Thurston's numerical maps)
- map the sites in $\mathcal{G}'$ to cartesian positions in $\mathbf{E^3}$ using the Weierstrass-Enneper equations for minimal surfaces
- reorient and scale the cartesian positions to match them with crystal coordinates of an asymmetric domain of the surface in the space group corresponding to $\mathcal{O}'$.
- determine crystal coords of all edges in *h-net* within a single $\mathcal{O}'$ domain
- extend to form 3-periodic euclidean structure using euclidean rotation and translation operations corresponding to cone points, translations in the orbifold



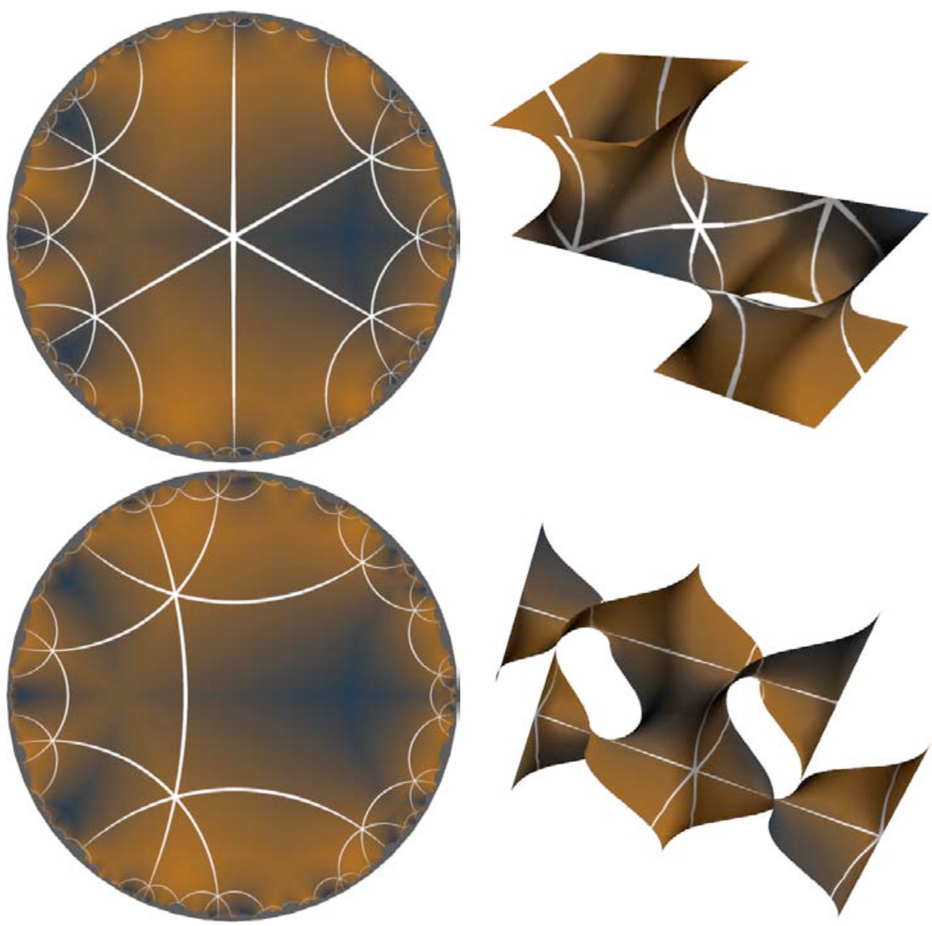
- A Platonic tiling of H^2 with six hyperbolic squares per vertex, $\{4,6\}$.

- The resulting $h-$net is drawn in the Poincaré disc model of H^2, shaded to reveal the kaleidoscopic symmetry of the tiling.

- Surface reticulation of the $\{4,6\}$ $h-net$ onto a fragment of the P three-periodic minimal surface.
  The resulting $e-$net

- its barycentric embedding, the $s-$net labelled $scq1$, identical to pcu.

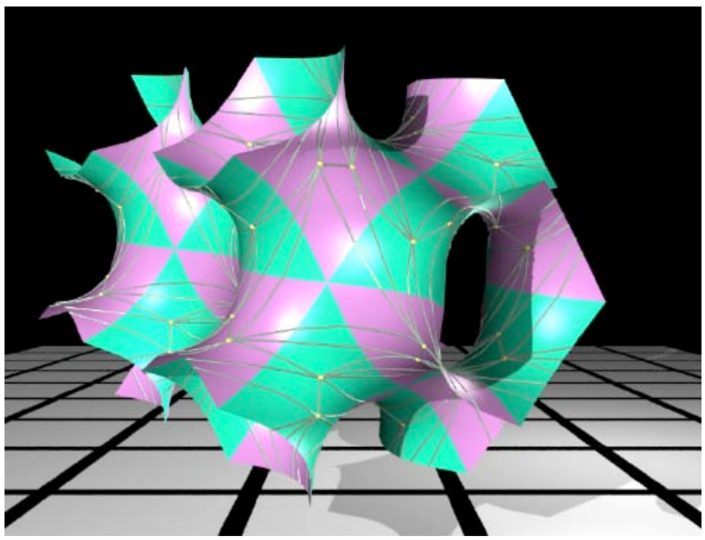
$h - net \rightarrow s - net$ is many-to-one:



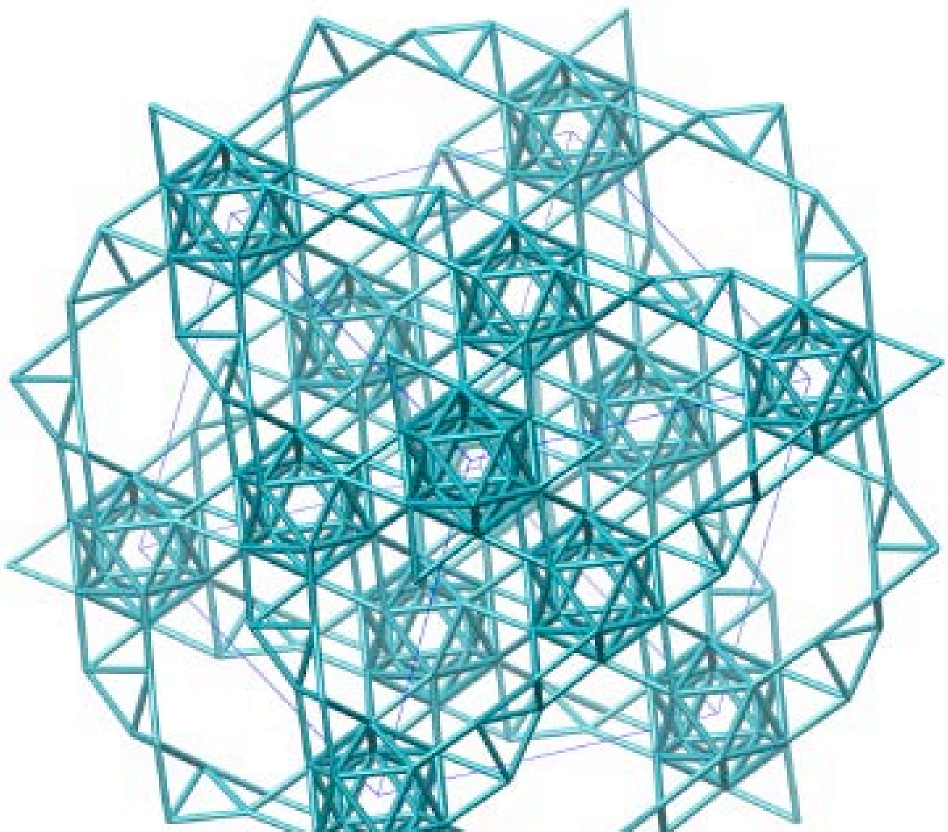
- Wrapping of (4.8.8.4.8.8) tilings of H^2 to form the pcu net
- Alternative locations of the $h-$net in H^2, both with orbifold symmetry $*4444$
- Wrapping onto the P and D surfaces, to give tetragonal and orthorhombic surface reticulations ($e-$nets)

Most symmetric crystallographic sub-group of optimal hyperbolic orbifold $*237$ has $Stellate$ orbifold 2223.

This wraps on the P and D to give:



The D embedding relaxes to the low density sphere packing made of regular icosahedra and octahedra:



Key references

- A comprehensive list of relevant references can be found on the epinet website.
- Good general texts are those by Thurston, Coxeter and Stillwell.
- A very easy introduction to manifolds and geometry is Weeks' book.
- An excellent introduction to, "Geometry and the Imagination in Minneapolis",covering orbifolds, non-euclidean geometry and topology can be obtained in an abbreviated form from the web. (Unfortunately, potentially confusing (now outdated) notation is used for handles and cross-caps in these notes. Handles and cross-caps are conventionally denoted by $\circ$ and $\times$ respectively; the Minneapolis notes use $\bullet$ and $\circ$).

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John Conway, Peter Doyle, Jane Gilman, and Bill Thurston. http://www.geon
1991.

H.S.M. Coxeter. *Introduction to Geometry*. Wiley, 1969.

S.J. Ramsden, V. Robins, S. Hungerford, and S.T. Hyde. epinet.anu.edu.au/m
2005. release 1.

Doris Schattschneider. *M.C. Escher, Visins of Symmetry*. Thames and Hudson, revised edition, 2004.

J. Stillwell. *Geometry of Surfaces*. Springer, 1995.

William P. Thurston. *Three-Dimensional Geometry and Topology, Volume 1*. Princeton University Press, 1997.

J. Weeks. *The Shape of Space*. Marcel Dekker, 1985.