Coincidence site lattices

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Coincidence Site Modules

Brief historical overview

1911: Friedel – *Leçons de Cristallographie*

1949: Kronberg, Wilson

mid sixties: CSLs: Ranganathan, Bollmann, Grimmer, . . .

mid ninties: quasicrystals → CSM

Baake, Pleasants, Warrington, . . .

2002: Sloane, Beferull–Lozano: *Quantizing Using Lattice Intersections*

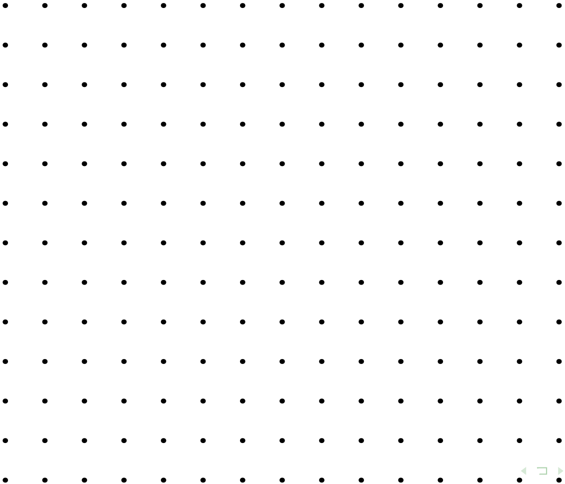
2005: Zou: Cartan-Dieudonné

1997-present: Aragón, Rodriguez et.al.: Clifford algebras

20xy: Baake, Grimm, Heuer, Moody, Pleasants, Scharlau, Loquias,

Glied, Huck, Dümke, PZ,. . .

Example



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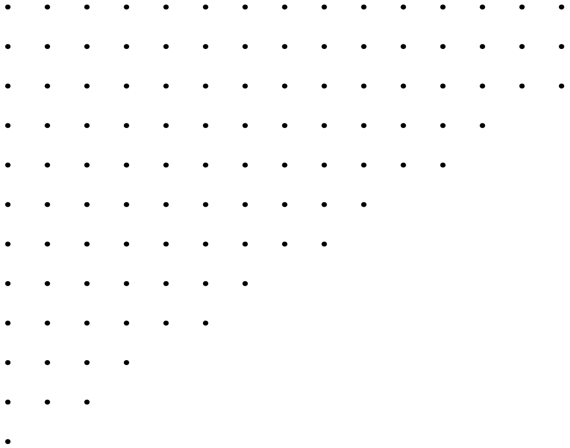

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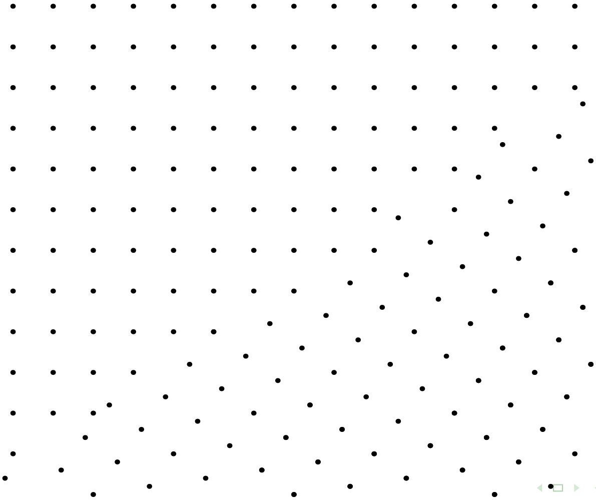

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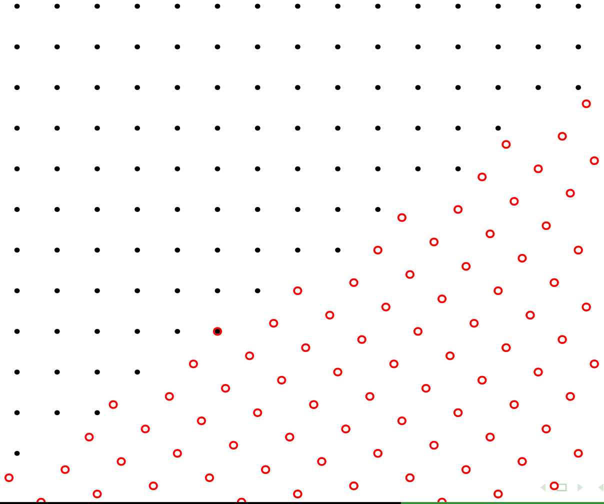

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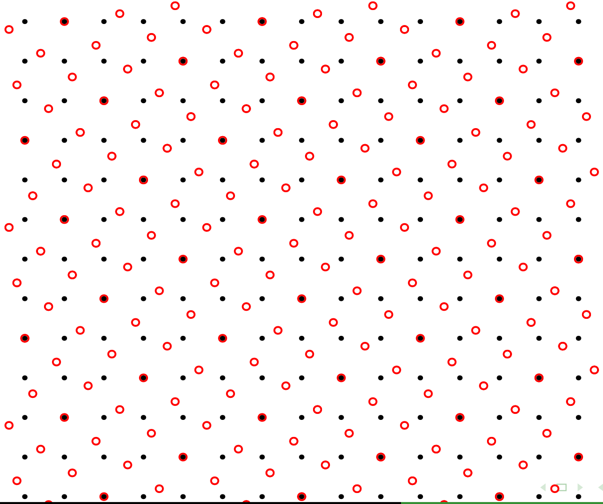

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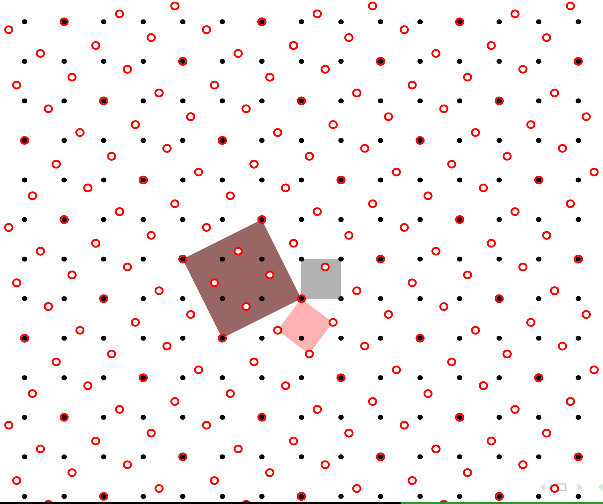


Modules and Lattices

- module M:

  $M =: \langle t_1, \ldots, t_k \rangle_{\mathbb{Z}} = \{ n_1 t_1 + \ldots + n_k t_k \} \subseteq \mathbb{R}^d$

  with $t_1, \ldots, t_k \in \mathbb{R}^d$ rationally independent,

  $\langle t_1, \ldots, t_k \rangle_{\mathbb{R}} = \mathbb{R}^d$, $k \geq d$
- lattice $\Gamma :=$ module with $k = d$
- submodule $M_1 \subseteq M$: full rank $k \iff [M : M_1]$ is finite.

Commensurate Modules

Lemma

The following are equivalent:

- M_1 *and* M_2 *are commensurate.*
- $M_1 \cap M_2$ *is a submodule of both* M_1 *and* M_2.
- $M_1 \cap M_2$ *is a submodule of* M_1 *or* M_2.
- *There exists an* $m \in \mathbb{N}$ *such that* $mM_1 \subseteq M_2$ *and* $mM_2 \subseteq M_1$.
- *There exists an* $m \in \mathbb{N}$ *such that* $mM_1 \subseteq M_2$ *or* $mM_2 \subseteq M_1$.

Ordinary CSMs

Definition

Let $M \subset \mathbb{R}^d$ be a module, $R \in O(d)$. Then

$$M(R) := M \cap RM$$

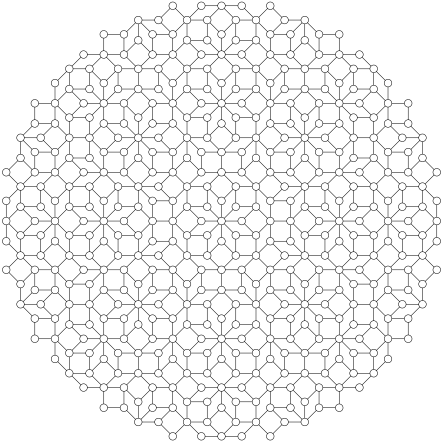
is called a (simple,ordinary) *coincidence site module* (CSM),

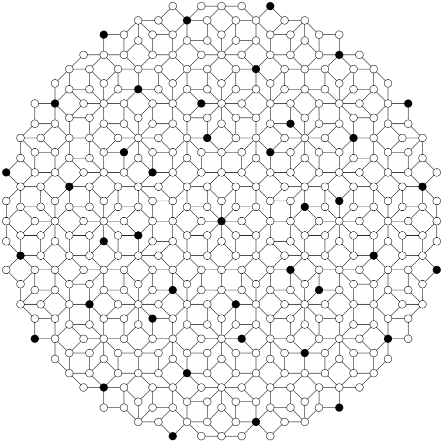
if M and RM are commensurate. The index

$$\Sigma_M(R) := [M : M(R)] < \infty$$

is called *coincidence index*.

Example: Ammann-Beenker tiling





R the rotation about the center by $\theta = \tan^{-1}\left(-2\sqrt{2}\right) \approx 109.5^\circ$, $\Sigma(R) = 9$

Coincidence isometries

Lemma

The set of all coincidence isometries

$$OC(M) := \{R \in O(d) \big| \Sigma_M(R) < \infty\}$$

forms a group, a subgroup of $O(d)$.

Ordinary CSLs

If $M = \Gamma$ then

$$\Sigma_\Gamma(R) = \frac{\mathrm{vol}(\Gamma(R))}{\mathrm{vol}(\Gamma)} = \frac{\mathrm{dens}(\Gamma)}{\mathrm{dens}(\Gamma(R))}$$

$$OC(\Gamma) = OC(\Gamma^*)$$

$$\Sigma_\Gamma(R) = \Sigma_{\Gamma^*}(R)$$

Symmetry Operations

Lemma

The following are equivalent:

1. $R \in P(M)$
2. $\Sigma_M(R) = 1$
3. $\mathrm{den}(R) = 1$.

Corollary

$$P(M) = \{R \in OC(M) \big| \Sigma_M(R) = 1\} \subseteq OC(M)$$

Equal CSMs

Lemma

$$S \in P(M) \quad \Longrightarrow \quad M(R) = M(RS)$$

But: $S \in P(M) \quad \not\Longleftarrow \quad M(R) = M(RS)$

Counterexamples

$\Gamma = (2\mathbb{Z})^2 \times \mathbb{Z},\ \mathbb{Z}^4,\ D_4,\ A_4$

Open question

When does $M(R) = M(RS)$ imply $S \in P(M)$?

Equal CSMs

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Open question

When does $M(R) = M(RS)$ imply $S \in P(M)$?

Root lattice A_4

$$\Phi^{rot}_{A_4}(s) = \frac{1+5^{1-s}}{1-5^{2-s}} \prod_{p\equiv\pm1(5)} \frac{(1+p^{-s})(1+p^{1-s})}{(1-p^{1-s})(1-p^{2-s})} \prod_{p\equiv\pm2(5)} \frac{1+p^{-s}}{1-p^{2-s}}$$

$$= 1 + \frac{5}{2^s} + \frac{10}{3^s} + \frac{20}{4^s} + \frac{30}{5^s} + \frac{50}{6^s} + \frac{50}{7^s} + \frac{80}{8^s} + \frac{90}{9^s} + \frac{150}{10^s} + \frac{144}{11^s} + \cdots$$

$$\Phi_{A_4}(s) = \left(1+6\,\tfrac{5^{-s}}{1-5^{2-s}}\right) \prod_{p\equiv\pm2(5)} \tfrac{1+p^{-s}}{1-p^{2-s}} \prod_{p\equiv\pm1(5)} \tfrac{1+p^{-s}+2p^{1-s}+2p^{-2s}+p^{1-2s}+p^{1-3s}}{(1-p^{2-s})(1-p^{1-2s})}$$

$$= 1 + \frac{5}{2^s} + \frac{10}{3^s} + \frac{20}{4^s} + \frac{6}{5^s} + \frac{50}{6^s} + \frac{50}{7^s} + \frac{80}{8^s} + \frac{90}{9^s} + \frac{30}{10^s} + \frac{144}{11^s} + \cdots$$

Properties of the Coincidence Index

Assume

- $M = \Gamma$
- M satisfies $[M : M(R)] = [RM : M(R)]$ for all R

Lemma

For any coincidence isometry R

$$\Sigma_M(R) = \Sigma_M(R^{-1}).$$

Coincidence Index of Products

Lemma

$\Sigma(R_1 R_2)$ divides $\Sigma(R_1)\Sigma(R_2)$.

Lemma

If $\Sigma(R_1)$ and $\Sigma(R_2)$ are coprime, then

$$\Sigma(R_1 R_2) = \Sigma(R_1)\Sigma(R_2).$$

Coincidences of Sublattices

Lemma

Let $M_1 \subseteq M$ *with index* $m := [M : M_1]$. *Then*

$$OC(M_1) = OC(M).$$

Let $\Sigma_1(R)$ *be the coincidence index with respect to* M_1. *Then*

$$\Sigma(R) \,\big|\, m\Sigma_1(R)$$
$$\Sigma_1(R) \,\big|\, m\Sigma(R).$$

Square lattice

Example: square lattice — Gaussian integers

Gaussian integers

$\Gamma = \{m + ni | m, n \in \mathbb{Z}\} = \mathbb{Z}[i]$

rotations

multiplication by a unimodular number $e^{i\varphi} \in \mathbb{C}$

coincidence rotations

$e^{i\varphi} = q + ir$, $q, r \in \mathbb{Q}$

$e^{i\varphi} = q + ir = \varepsilon \frac{m+ni}{m-ni}$

We want a reduced fraction!!! $\longrightarrow$

We want some kind of prime factorization.

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Prime factorization in $\mathbb{N}$

unit

1 is the only integer n whose inverse n^{-1} is in $\mathbb{N}$.

primes

p is a prime number if it is not a unit and cannot be written as a product of two non–units.

$2, 3, 5, 7, 11, 13, \ldots$

prime factorization is unique up to permutations

$6 = 2 \cdot 3 = 3 \cdot 2$

Prime factorization in $\mathbb{Z}$

units

± 1 are the only integers n whose inverse n^{-1} is in $\mathbb{Z}$.

primes

$p \neq 0$ is a prime number if it is not a unit and cannot be written as a product of two non–units.

$\pm 2, \pm 3, \pm 5, \pm 7, \pm 11, \pm 13, \ldots$

prime factorization is unique up to permutations and units

$6 = 2 \cdot 3 = 3 \cdot 2 = (-2)(-3) = (-1)(-2)3$

Prime factorization in $\mathbb{Z}[i]$

units

$1, -1, i, -i$

primes

$2 = (1+i)(1-i) = -i(1+i)^2$ (“ramifying prime”)

$p \equiv 3 \pmod 4$ (“inert primes”)

$p \equiv 1 \pmod 4 \Rightarrow p = \omega\bar{\omega}_p, \omega_p = m + in \in \mathbb{Z}[i]$ (“splitting primes”)

prime factorization is unique up to permutations and units

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Coincidence rotations of $\mathbb{Z}[i]$

coincidence rotations

$$e^{i\varphi} = \varepsilon \frac{z}{\bar{z}} = \varepsilon \prod_{p \equiv 1\,(4)} \left(\frac{\omega_p}{\bar{\omega}_p} \right)^{n_p}$$

ε unit, only finitely many $n_p \neq 0$

coincidence index

$$\Sigma(e^{i\varphi}) = \prod_{p \equiv 1\,(4)} p^{|n_p|}$$

spectrum

set of all integers that contain only prime factors $p \equiv 1 \pmod 4$.

CSLs of $\mathbb{Z}[i]$

$$\omega(\varphi) := \prod_{\substack{p\equiv 1\,(4)\\ n_p>0}} \omega_p^{n_p} \prod_{\substack{p\equiv 1\,(4)\\ n_p<0}} \bar{\omega}_p^{n_p}$$

CSLs

$$\mathbb{Z}[i] \cap e^{i\varphi}\mathbb{Z}[i] = \omega(\varphi)\mathbb{Z}[i]$$

Number of different CSLs and coincidence rotations

number of CSLs: $f(\Sigma)$

number of coincidence rotations: $4f(\Sigma) = 4f^{rot}(\Sigma)$

$$\begin{aligned} f(1) &= 1 \\ f(p^r) &= 2 \quad \text{if } p \equiv 1 \pmod 4 \\ f(p^r) &= 0 \quad \text{if } p \not\equiv 1 \pmod 4 \\ f(mn) &= f(m)f(n) \quad \text{if } \gcd(m,n) = 1 \end{aligned}$$

Generating function — Dirichlet series

$$\begin{aligned}
\Phi(s) &= \sum_{m=1}^{\infty} \frac{f(m)}{m^s} = \prod_{p\equiv 1(4)} \frac{1+p^{-s}}{1-p^{-s}} \\
&= \frac{1}{1+2^{-s}} \frac{\zeta_{\mathbb{Q}(i)}(s)}{\zeta(2s)} \\
&= \frac{1}{1+2^{-s}} \frac{L(s,\chi_{-4})\zeta(s)}{\zeta(2s)} \\
&= 1 + \frac{2}{5^s} + \frac{2}{13^s} + \frac{2}{17^s} + \frac{2}{25^s} + \frac{2}{29^s} + \frac{2}{37^s} + \frac{2}{41^s} \\
&\quad + \frac{2}{53^s} + \frac{2}{61^s} + \frac{4}{65^s} + \frac{2}{73^s} + \dots
\end{aligned}$$

Zeta functions and L–series

Riemann Zeta–function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta_{\mathbb{Q}}(s)$$

Zeta–function of $\mathbb{Q}(i)$

$$\zeta_{\mathbb{Q}(i)}(s) = L(s, \chi_{-4})\zeta(s) = \frac{1}{1-2^{-s}} \prod_{p\equiv 1(4)} \frac{1}{(1-p^{-s})^2} \prod_{p\equiv 3(4)} \frac{1}{1-p^{-2s}}$$

L–series

$$L(s, \chi_{-4}) = \sum_{n=1}^{\infty} \frac{\chi_{-4}(n)}{n^s}$$

Dirichlet character

$$\chi_{-4}(n) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \equiv 1 \bmod 4, \\ -1, & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

$$\chi_{-4}(mn) = \chi_{-4}(m)\chi_{-4}(n) \text{ if } m, n \text{ coprime}$$

$$\chi_{-4}(n) = \chi_{-4}(n+4)$$

Theorem of Delange (simplified version)

Let

$$F(s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}=g(s)+h(s)/(s-\alpha)^{n+1}$$

Dirichlet series with nonnegative coefficients

converging for $s>\alpha>0$,

$g(s), h(s)$ holomorhic for $\mathrm{Re}(s)\geq\alpha$

Then

$$A(x):=\sum_{m\leq x}a(m)\sim\frac{h(\alpha)}{\alpha\cdot n!}x^{\alpha}(\log(x))^{n}$$

Growth rate square lattice

$f(m)$ number of CSLs of square lattice

$$\sum_{m\leq x} f(m) \sim \frac{1}{\pi} x$$

Cubic lattices

Quaternions – basic definitions

Aim: turn $\mathbb{R}^4$ into algebra

basis:

$$\mathbf{e} = (1,0,0,0) \quad \mathbf{i} = (0,1,0,0) \quad \mathbf{j} = (0,0,1,0) \quad \mathbf{k} = (0,0,0,1)$$

defining products:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{e}$$

further products:

$$\mathbf{ijk} = -\mathbf{kji}$$

Quaternions – basic definitions

$\mathbb{H} = \mathbb{H}(\mathbb{R}) = \mathbb{R}\mathbf{e} + \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ is a non–commutative assiciative division algebra

conjugation

$$\bar{\mathbf{q}} = \kappa\mathbf{e} - \lambda\mathbf{i} - \mu\mathbf{j} - \nu\mathbf{k} \quad \text{where} \quad \mathbf{q} = \kappa\mathbf{e} + \lambda\mathbf{i} + \mu\mathbf{j} + \nu\mathbf{k}$$

real- and imaginary part

$$\mathrm{Im}(\mathbf{q}) = \kappa\mathbf{e} \qquad \mathrm{Re}(\mathbf{q}) = \lambda\mathbf{i} + \mu\mathbf{j} + \nu\mathbf{k}$$

Remark

$$\mathrm{Im}(\mathbb{H}(\mathbb{R})) \simeq \mathbb{R}^3$$

Quaternions – basic definitions

norm

$$n(\mathbf{q}) = |\mathbf{q}|^2 = \mathbf{q}\bar{\mathbf{q}} = \kappa^2 + \lambda^2 + \mu^2 + \nu^2$$

inverse

$$\mathbf{q}^{-1} = \frac{1}{n(\mathbf{q})}\bar{\mathbf{q}} = \frac{1}{|\mathbf{q}|^2}\bar{\mathbf{q}}$$

trace

$$\mathrm{tr}(\mathbf{q}) = \mathbf{q} + \bar{\mathbf{q}} = 2\,\mathrm{Re}(\mathbf{q})$$

inner product

$$\langle \mathbf{q}_1, \mathbf{q}_2 \rangle = \mathrm{tr}(\mathbf{q}_1\bar{\mathbf{q}}_2) = \mathrm{tr}(\bar{\mathbf{q}}_1\mathbf{q}_2) = 2(\kappa_1\kappa_2 + \lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2)$$

Integral quaternions

integral quaternions

$$\mathbf{q} \text{ is integral } \iff n(\mathbf{q}), \mathrm{tr}(\mathbf{q}) \in \mathbb{Z}$$

Hurwitz quaternions

$$\mathbb{J} = \{\ell\mathbf{i} + m\mathbf{j} + n\mathbf{k} + \frac{k}{2}(\pm 1, \pm 1, \pm 1, \pm 1) | k, \ell, m, n \in \mathbb{Z}\}$$

units

$$\pm\mathbf{e}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}, \frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$$

Integral quaternions

Lipschitz quaternions

$$\mathbb{L} = \{\kappa\mathbf{e} + \lambda\mathbf{i} + \mu\mathbf{j} + \nu\mathbf{k} | \kappa, \lambda, \mu, \nu \in \mathbb{Z}\}$$

primitive quaternion

$\mathbf{q}$ primitive $\iff$ $\mathbf{q}$ is Lipschitz and $\gcd(\kappa, \lambda, \mu, \nu) = 1$

Prime factorization

- prime factorization in $\mathbb{J}$ is essentially unique for primitive quaternions
- for any prime $p \in \mathbb{N}$ there exists a $\mathbf{p}$ such that $p = \mathbf{p}\bar{\mathbf{p}}$
- for any odd prime p there are $p+1$ non associate prime quaternions $\mathbf{p}$

Quaternions and rotations

$$\mathbb{R}^3 \simeq \mathrm{Im}(\mathbb{H})$$

$$x \longleftrightarrow \mathbf{x} = (0, x)$$

rotation

$$R(\mathbf{q})x = \mathbf{q}\mathbf{x}\mathbf{q}^{-1} = \frac{1}{|\mathbf{q}|^2}\mathbf{q}\mathbf{x}\bar{\mathbf{q}}$$

rotoreflections

$$S(\mathbf{q})x = \mathbf{q}\bar{\mathbf{x}}\mathbf{q}^{-1} = \frac{1}{|\mathbf{q}|^2}\mathbf{q}\bar{\mathbf{x}}\bar{\mathbf{q}}$$

Quaternions and rotations

$$R(\mathbf{q}) = \frac{1}{|\mathbf{q}|^2} \left(\begin{array}{ccc} \kappa^2+\lambda^2-\mu^2-\nu^2 & -2\kappa\nu+2\lambda\mu & 2\kappa\mu+2\lambda\nu \\ 2\kappa\nu+2\lambda\mu & \kappa^2-\lambda^2+\mu^2-\nu^2 & -2\kappa\lambda+2\mu\nu \\ -2\kappa\mu+2\lambda\nu & 2\kappa\lambda+2\mu\nu & \kappa^2-\lambda^2-\mu^2+\nu \end{array} \right.$$

rotation axis:

$$(\lambda, \mu, \nu) = \mathrm{Im}(\mathbf{q})$$

rotation angle:

$$\cos(\phi) = \frac{\kappa^2-\lambda^2-\mu^2-\nu^2}{\kappa^2+\lambda^2+\mu^2+\nu^2}$$

Quaternions and rotations

- quaternions $\mathbf{q}$ with $|\mathbf{q}|^2 = 1$ form a group, a double cover of $SO(3)$
- unit quaternions $+ \frac{1}{\sqrt{2}}(\pm 1, \pm 1, 0, 0)$ + permutations form a group: double cover of O
- $SO(3, \mathbb{Q})$ can be parametrized by primitive quaternions

Coincidences of cubic lattices

primitive cubic lattice: $\Gamma_{pc} = \mathrm{Im}(\mathbb{L}) \simeq \mathbb{Z}^3$

body centered cubic lattice: $\Gamma_{bcc} = \mathrm{Im}(\mathbb{J})$

face centered cubic lattice: $\Gamma_{fcc} = \bigcup_{i=0}^{3} x_i + \Gamma_{pc}$ with

$x_0 = 0, x_1 = (0,1,1), x_2 = (1,0,1), x_3 = (1,1,0)$

for all three lattices:

$OC(\Gamma) = O(3, \mathbb{Q})$

$SOC(\Gamma) = SO(3, \mathbb{Q})$

Cubic lattices — Coincidence index

Lemma

For all cubic lattices:

$$\Sigma(R) = \frac{|\mathbf{q}|^2}{2^\ell},$$

where 2^ℓ *maximal power that divides* $|\mathbf{q}|^2$*.*

Remark

- $\Sigma(R)$ is odd
- $\Sigma(R)$ runs over all positive odd integers

Cubic lattices — CSLs

Body centered cubic lattice

$$\Gamma_{bcc} = \mathrm{Im}(\mathbb{J})$$

$$\Gamma_{bcc}(R(\mathbf{q})) = \mathrm{Im}(\mathbf{q}\mathbb{J}) \quad \text{if } |\mathbf{q}|^2 \text{ odd}$$

Primitive cubic lattice

$$\Gamma_{pc} = \mathrm{Im}(\mathbb{H})$$

$$\Gamma_{pc}(R(\mathbf{q})) = \mathrm{Im}(\mathbf{q}\mathbb{H}) \quad \text{if } |\mathbf{q}|^2 \text{ odd}$$

Remark one–to–one correspondence

CSLs ↔ left ideals $\mathbf{q}\mathbb{J}$, $\mathbf{q}$ primitive, $|\mathbf{q}|^2$ odd

Cubic lattices — number of CSLs

number of different CSLs for fixed Σ:

$f(1) = 1$

$f(2m) = 0$

$f(p^r) = (p+1)p^{r-1}$

$f(mn) = f(m)f(n) \qquad \text{if } m, n \text{ are coprime}$

Remark: multiplicativity is a consequence of the essentially unique prime factorization in $\mathbb{J}$.

Cubic lattices — Dirichlet series

$$
\begin{aligned}
\Phi(s) &= \sum_{m=1}^{\infty} \frac{f(m)}{m^s} = \prod_{p\neq 2} \frac{1+p^{-s}}{1-p^{1-s}} = \\
&= \frac{1-2^{1-s}}{1+2^{-s}} \frac{\zeta(s)\zeta(s-1)}{\zeta(2s)} = \\
&= 1 + \frac{4}{3^s} + \frac{6}{5^s} + \frac{8}{7^s} + \frac{12}{9^s} + \frac{12}{11^s} + \frac{14}{13^s} + \frac{24}{15^s} + \frac{18}{17^s} + \dots
\end{aligned}
$$

Known CSLs

- Square lattice, hexagonal lattice
- certain planar modules with N–fold symmetry
- certain planar lattices (maximal and non–maximal orders)
- cubic lattices and related modules
- hypercubic lattices
- A_4–lattice, ring of icosians

Number of coincidence isometries

$$f^{iso}(m) = \frac{\text{number of coincidence isometries of index } m}{|P|}$$

Theorem

f^{iso} is a supermultiplicative function, i.e.

$$f^{iso}(mn) \geq f^{iso}(m) f^{iso}(n)$$

if m, n are coprime.

Number of CSLs

$f(m) =$ number of (simple) CSLs of index m

Theorem

f is a supermultiplicative function, i.e.

$$f(mn) \geq f(m)f(n)$$

if m, n are coprime.

(Non–)Multiplicativity

f and f^{iso} are multiplicative for

- square lattice, hexagonal lattice, various modules with n–fold symmetry
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- hypercubic lattices, A_4, icosian ring

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Number of CSLs for related lattices

Lemma

If Γ_1 *and* Γ_2 *are similar, then*

$$f_{\Gamma_1}(m) = f_{\Gamma_2}(m).$$

Moreover

$$f_{\Gamma^*}(m) = f_{\Gamma}(m).$$

Similar Sublattices

Similarity Transformations

Definition

Let $\alpha \in \mathbb{R}^+$ and $R \in O(d)$. Then

$$\begin{aligned} A\colon \mathbb{R}^d &\to \mathbb{R}^d \\ x &\to \alpha R x \end{aligned}$$

is called a linear similarity transformation.

Similar Sublattice

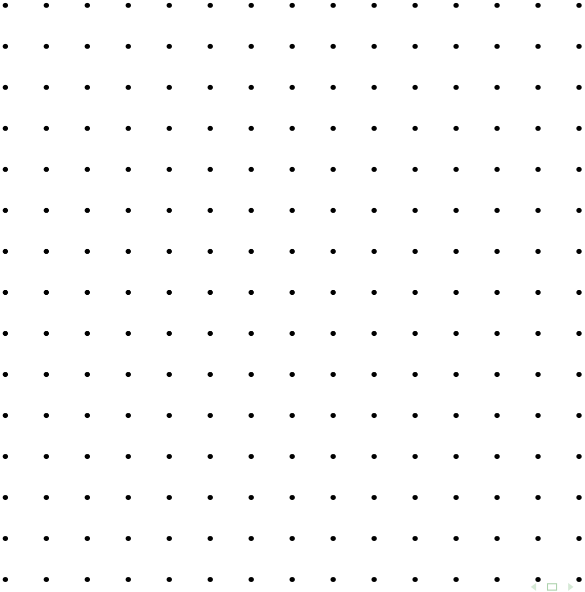
Definition

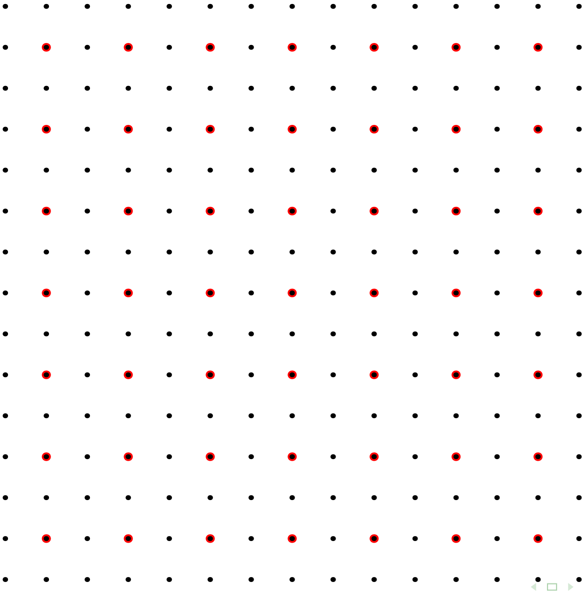
Let $A = \alpha R$ be a linear similarity transformation and $\Gamma \subseteq \mathbb{R}^d$ a lattice.

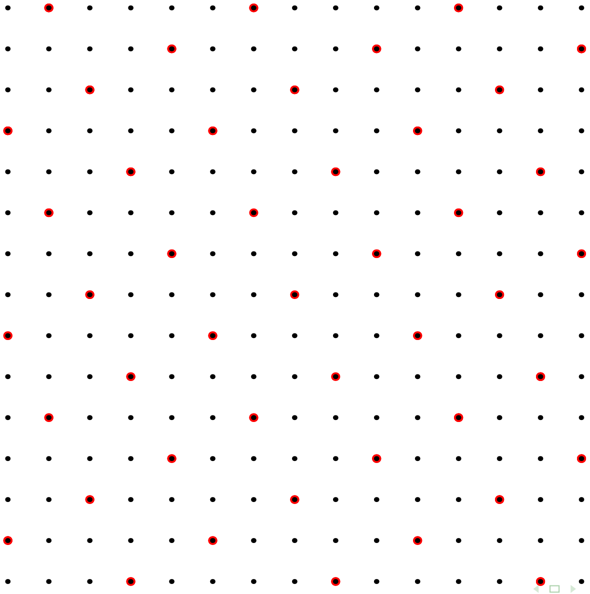
Then A is called a similarity transformation of Γ if

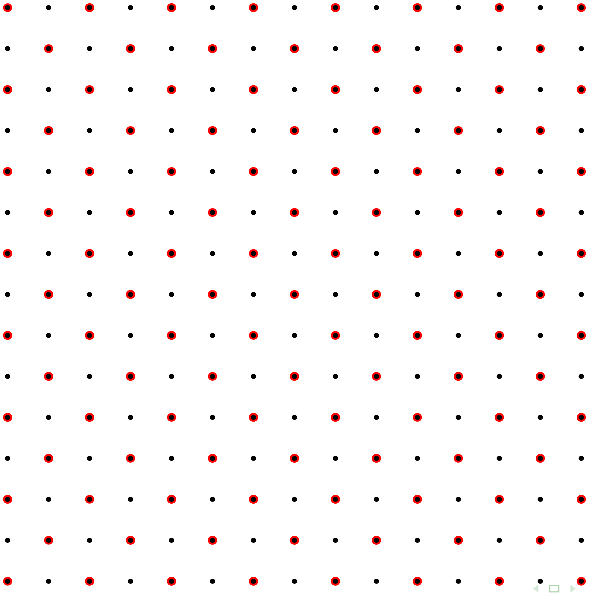
$$A\Gamma = \alpha R\Gamma \subseteq \Gamma.$$

In this case $A\Gamma = \alpha R\Gamma$ is called a *similar sublattice (similarity sublattice)*.









Examples: Trivial Similar Sublattices and Similarity Transformations

- $\Gamma \subseteq \Gamma$
- $R\Gamma \subseteq \Gamma$, if $R \in P(\Gamma) := \{R \in O(d) \big| R\Gamma = \Gamma\}$
- $n\Gamma \subseteq \Gamma$, $n \in \mathbb{N}$

Index of a Similar Sublattice

Lemma

For any similar sublattice of the lattice $\Gamma \subseteq \mathbb{R}^d$*:*

$$[\Gamma : \alpha R\Gamma] = \alpha^d \in \mathbb{N}.$$

Example – square lattice

number of SSLs

$$\Psi_{\mathbb{Z}^2}(s) = \zeta_{\mathbb{Q}(i)}(s) = \frac{1}{1-2^{-s}} \prod_{p\equiv 1(4)} \frac{1}{(1-p^{-s})^2} \prod_{p\equiv 3(4)} \frac{1}{1-p^{-2s}}$$
$$= 1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{2}{5^s} + \frac{1}{8^s} + \frac{1}{9^s} + \frac{2}{10^s} + \frac{2}{13^s} + \frac{1}{16^s}$$
$$+ \frac{2}{17^s} + \frac{1}{18^s} + \frac{2}{20^s} + \frac{2}{25^s} + \frac{2}{26^s} + \frac{2}{29^s} + \frac{1}{32^s} + \dots$$

Primitive similar sublattices

Definition

A similar sublattice Γ_1 of Γ is called primitive,

if $\frac{1}{n}\Gamma_1 \not\subseteq \Gamma$ for all $n > 1$.

Lemma

A similar sublattice Γ_1 of Γ is primitive if and only if there exists an $R \in OS(\Gamma)$ such that

$$\Gamma_1 = \mathrm{den}_{\Gamma}(R) R \Gamma .$$

Primitive SSLs of the Square lattice

$$\Psi^{pr}_{\mathbb{Z}^2}(s) = \frac{\Psi_{\mathbb{Z}^2}(s)}{\zeta(2s)} = (1+2^{-s}) \prod_{p\equiv 1(4)} \frac{1+p^{-s}}{1-p^{-s}}$$

$$= 1 + \frac{1}{2^s} + \frac{2}{5^s} + \frac{2}{10^s} + \frac{2}{13^s} + \frac{2}{17^s} + \frac{2}{25^s} + \frac{2}{26^s} + \dots$$

$$\Psi_{\mathbb{Z}^2}(s) = \zeta_{\mathbb{Q}(i)}(s) = \zeta(2s)\Psi^{pr}_{\mathbb{Z}^2}(s)$$

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Similarity Isometries

Definition

An isometry $R \in O(d)$ is called a *similarity isometry* of Γ, if there exists an $\alpha \in \mathbb{R}^+$ such that αR is a similarity transformation of Γ.

Lemma

The set of all similarity isometries of Γ *forms a group, called* $OS(\Gamma)$*. In particular* $OS(\Gamma)$ *is a countable subgroup of* $O(d)$*.*

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Similarity Isometries for related Lattices

Lemma

Let Γ_1 *and* Γ_2 *be commensurate. Then*

$$OS(\Gamma_1) = OS(\Gamma_2).$$

Moreover

$$OS(\alpha R\Gamma) = R\,OS(\Gamma)R^{-1}$$
$$OS(\Gamma) = OS(\Gamma^*).$$

Denominator (“Minimal Blow–up factor”)

Definition

Let $R \in OS(\Gamma)$. Then

$$\mathrm{den}_{\Gamma}(R) := \min\{\alpha \in \mathbb{R}^{+} \big| \alpha R\Gamma \subseteq \Gamma\}.$$

Denominator — Example

Example

Square lattice $\Gamma = \mathbb{Z}^2$

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \in OS(\Gamma)$$

$$\mathrm{den}_\Gamma(R) = \sqrt{2}$$

Denominator ("Minimal Blow–up factor")

Lemma

Let $R \in OS(\Gamma)$. Then

$$\{\alpha \in \mathbb{R} \big| \alpha R\Gamma \subseteq \Gamma\} = \mathrm{den}_{\Gamma}(R)\mathbb{Z}$$

$$\mathrm{den}_{\Gamma}(R)^d \in \mathbb{N}.$$

Denominator of the Inverse Isometry

Lemma

Let $R \in OS(\Gamma)$. *Then*

$$\frac{\mathrm{den}_\Gamma(R)^{d-1}}{\mathrm{den}_\Gamma(R^{-1})} \in \mathbb{N}$$

$$\mathrm{den}_\Gamma(R)^{d-1}\,\mathrm{den}_\Gamma(R^{-1}) \in \mathbb{N}$$

Corollary

If $d = 2$ *then*

$$\mathrm{den}_\Gamma(R^{-1}) = \mathrm{den}_\Gamma(R).$$

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Example – Unequal denominators

Example

$$\Gamma = \mathbb{Z} \times (\xi\mathbb{Z}) \times \cdots \times (\xi^{d-1}\mathbb{Z})$$

$$R\mathbf{e}_i = \mathbf{e}_{i+1}$$

$$\mathrm{den}_\Gamma(R) = \xi$$

$$\mathrm{den}_\Gamma(R^{-1}) = \xi^{d-1}$$

Denominator of related lattices

Lemma

If $\Gamma_2 \subseteq \Gamma_1$ *with* $[\Gamma_1 : \Gamma_2] = m$ *then*

$$m\frac{\mathrm{den}_1(R)}{\mathrm{den}_2(R)} \in \mathbb{N} \qquad \textit{and} \qquad m\frac{\mathrm{den}_2(R)}{\mathrm{den}_1(R)} \in \mathbb{N}.$$

Moreover

$$\mathrm{den}_{\Gamma^*}(R) = \mathrm{den}_{\Gamma}(R^{-1}).$$

Coincidence Isometries versus Similarity isometries

Lemma

(S. Glied, 2008)

- $OC(\Gamma) \subseteq OS(\Gamma)$
- $OS(\Gamma)/OC(\Gamma)$ *is abelian.*
- $g^d = e$ *for any* $g \in OS(\Gamma)/OC(\Gamma)$.
- *In particular, if d=p, then* $OS(\Gamma)/OC(\Gamma)$ *is a p–group.*

Coincidence Isometries versus Similarity isometries

Lemma

$$OC(\Gamma) = \{R \in OS(\Gamma) \big| \, \mathrm{den}(R) \in \mathbb{N}\} \subseteq OS(\Gamma) \subset O(d)$$

Coincidence Index and Denominator

Lemma

Let $m := \operatorname{lcm}(\operatorname{den}_\Gamma(R), \operatorname{den}_\Gamma(R^{-1}))$ *and* $n := \gcd(\operatorname{den}_\Gamma(R), \operatorname{den}_\Gamma(R^{-1}))$. *Then*

$$m \mid \Sigma(R) \mid n^d \qquad \textit{and} \qquad \Sigma(R)^2 \mid m^d$$

Remark

If $d = 2$ then

$$\Sigma(R) = \operatorname{den}_\Gamma(R) = \operatorname{den}_\Gamma(R^{-1}).$$

Coincidence Index and Denominator

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$$m \,\big|\, \Sigma(R) \,\big|\, n^d \qquad \text{and} \qquad \Sigma(R)^2 \,\big|\, m^d$$

Remark

If $d = 2$ then

$$\Sigma(R) = \mathrm{den}_\Gamma(R) = \mathrm{den}_\Gamma(R^{-1}).$$

Number of Similar Sublattices

Definition

$a_\Gamma(m) =$ number of similar sublattices of index m

Theorem

a_Γ is a supermultiplicative function, i.e.

$$a_\Gamma(mn) \geq a_\Gamma(m)a_\Gamma(n)$$

if m, n are coprime.

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if m, n *are coprime.*

Dirichlet series

Definition

$$\Psi_\Gamma(s) := \sum_{m=1}^{\infty} \frac{a_\Gamma(m)}{m^s}$$

$$\Psi_\Gamma^{pr}(s) := \sum_{m=1}^{\infty} \frac{a_\Gamma^{pr}(m)}{m^s}$$

Lemma

$$\Psi_\Gamma(s) = \zeta(ds)\Psi_\Gamma^{pr}(s)$$

Affine Coincidences and Shifted Lattices

Affine Coincidences of Modules

Definition

Let $M \subset \mathbb{R}^d$ be a module, $R \in O(d)$, $v \in \mathbb{R}^d$. Then

$$M(v, R) := M \cap (v, R)M$$

is called an *affine coincidence site module* (CSM),

if $M(v, R)$ is an (affine) submodule of full rank.

(v, R) is called an affine coincidence isometry.

Affine Coincidences of Modules

Theorem

$$AC(M) = \{(v, R) : R \in OC(M) \textit{ and } v \in M + RM\}$$

Remark

$AC(M)$ is not a group in general.

Affine Coincidences of Lattices

Grimmer 1974

$$AC(\Gamma) = \{(v, R) : R \in OC(\Gamma) \text{ and } v \in \Gamma + R\Gamma\}$$

$\Gamma + R\Gamma$ . . . DSC lattice

Coincidences of shifted lattices

Linear coincidences of shifted lattices:

$$(x+\Gamma)\cap R(x+\Gamma)$$

Theorem

$$OC(x+\Gamma)=\{R\in OC(\Gamma) : Rx-x\in\Gamma+R\Gamma\}$$

- In general, $OC(x+\Gamma)$ is not a group.
- Problem: Product of coincidence isometries need not be a coincidence isometry

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- In general, $OC(x+\Gamma)$ is not a group.
- Problem: Product of coincidence isometries need not be a coincidence isometry

Coincidence isometries of $x + \mathbb{Z}[i]$

Theorem

Let $\Gamma = \mathbb{Z}[i]$ *and* $x \in \mathbb{C}$.

1. $SOC(x + \Gamma)$ *is a subgroup of* $SOC(\Gamma)$
2. $OC(x + \Gamma)$ *is a subgroup of* $OC(\Gamma)$ *if and only if for any* T_1, $T_2 \in OC(x + \Gamma) \setminus SOC(x + \Gamma)$, $T_1 T_2 \in SOC(x + \Gamma)$

Coincidence isometries of $x + \mathbb{Z}[i]$

- $x = \dfrac{r}{q}$ where $r, q \in \mathbb{Z}[i]$, r and q relatively prime

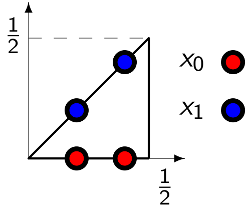
Lemma

$$SOC(x + \Gamma) = SOC\left(\tfrac{1}{q} + \Gamma\right)$$

Lemma

If q has no prime factor ω_p, then $OC(x + \Gamma)$ is a group.

Example: q=5



- $x_0 = \frac{1}{5}, \frac{2}{5}$ and $x_1 = \frac{1}{5} + \frac{1}{5}i, \frac{2}{5} + \frac{2}{5}i \Rightarrow q = 5$
- $SOC(x_0 + \Gamma) = SOC(x_1 + \Gamma) = SOC\left(\frac{1}{5} + \Gamma\right)$
- $OC(x_0 + \Gamma)$ and $OC(x_1 + \Gamma)$ are groups

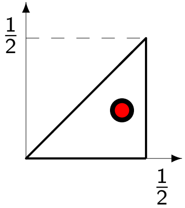
Example: q=5

number of CSLs of $x+\Gamma$

number of coincidence rotations of $x+\Gamma$

$$\begin{aligned}\Phi_{x+\Gamma}(s) &= \frac{1-5^{-s}}{1+5^{-s}}\Phi(s)\\ &= 1+\tfrac{2}{13^s}+\tfrac{2}{17^s}+\tfrac{2}{29^s}+\tfrac{2}{37^s}+\tfrac{2}{41^s}+\tfrac{2}{53^s}+\tfrac{2}{61^s}+\tfrac{2}{73^s}+\dots\\ \Phi(s) &= 1+\tfrac{2}{5^s}+\tfrac{2}{13^s}+\tfrac{2}{17^s}+\tfrac{2}{25^s}+\tfrac{2}{29^s}+\tfrac{2}{37^s}+\tfrac{2}{41^s}+\tfrac{2}{53^s}\\ &\quad+\tfrac{2}{61^s}+\tfrac{4}{65^s}+\tfrac{2}{73^s}+\dots\end{aligned}$$

Example: $q = 2 - i$



- $x = \frac{2}{5} + \frac{1}{5} i = \frac{1}{2-i} \Rightarrow q = 2 - i$
- $SOC(x+\Gamma) = SOC\left(\frac{1}{2-i} + \Gamma\right) = SOC\left(\frac{1}{5} + \Gamma\right)$
- $OC(x+\Gamma)$ is **NOT** a group!

Example: $q = 2 - i$

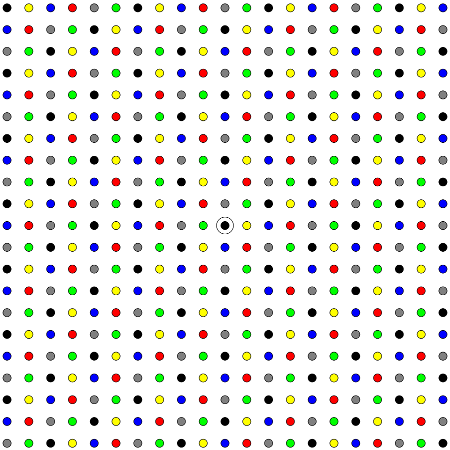
number of CSLs of $x + \Gamma$

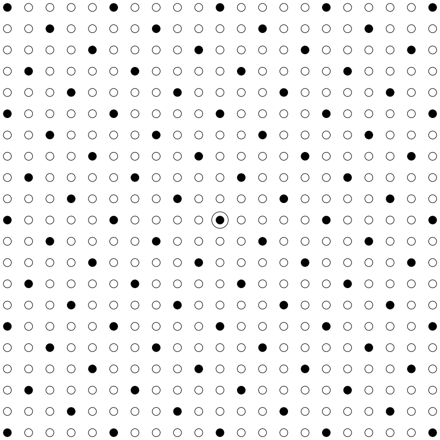
number of coincidence isometries of $x + \Gamma$

$$\begin{aligned}
\Phi_{x+\Gamma}(s) &= \frac{1 + 3 \cdot 5^{-s}}{1 + 5^{-s}} \Phi(s) \\
&= 1 + \tfrac{4}{5^s} + \tfrac{2}{13^s} + \tfrac{2}{17^s} + \tfrac{4}{25^s} + \tfrac{2}{29^s} + \tfrac{2}{37^s} + \tfrac{2}{41^s} + \tfrac{2}{53^s} \\
&\quad + \tfrac{2}{61^s} + \tfrac{8}{65^s} + \tfrac{2}{73^s} + \dots \\
\Phi(s) &= 1 + \tfrac{2}{5^s} + \tfrac{2}{13^s} + \tfrac{2}{17^s} + \tfrac{2}{25^s} + \tfrac{2}{29^s} + \tfrac{2}{37^s} + \tfrac{2}{41^s} + \tfrac{2}{53^s} \\
&\quad + \tfrac{2}{61^s} + \tfrac{4}{65^s} + \tfrac{2}{73^s} + \dots
\end{aligned}$$

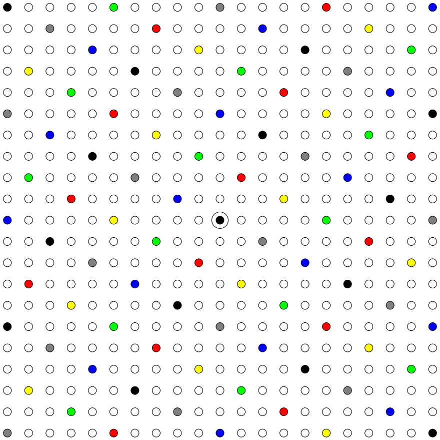
Coincidences of Colourings

Colourings



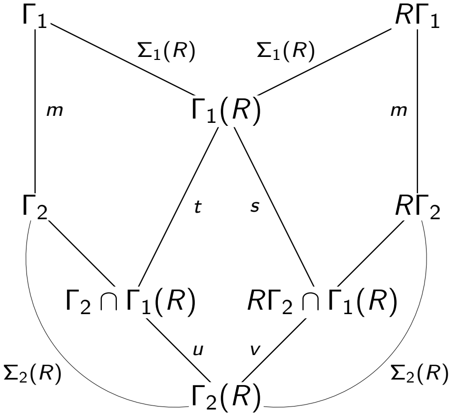


rotation about the origin (counterclockwise) by $\theta = \arctan\left(\frac{3}{4}\right)$



colouring of $\Gamma_1(R)$

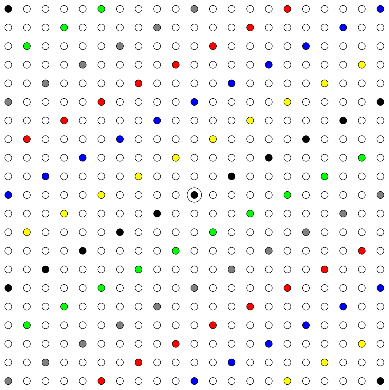
sublattice diagram



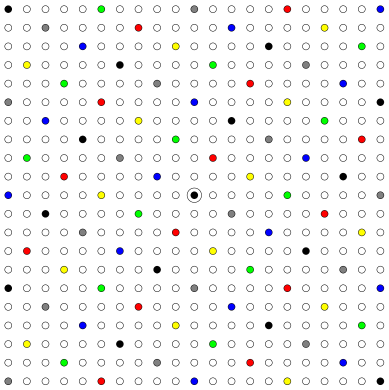
Theorem

$$\Sigma_2(R) = \frac{t \cdot u \cdot \Sigma_1(R)}{m} = \frac{s \cdot v \cdot \Sigma_1(R)}{m}$$

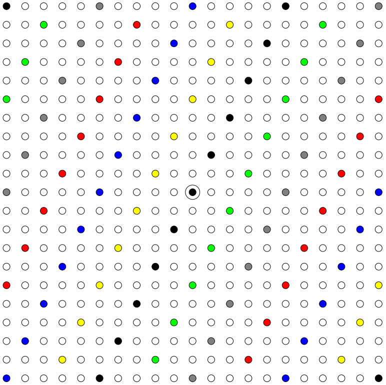
where $s, t, u, v \mid m$.



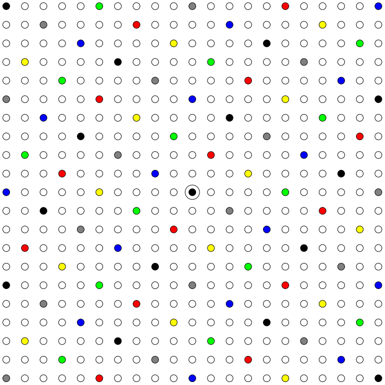
colouring of $\Gamma_1(R^{-1})$



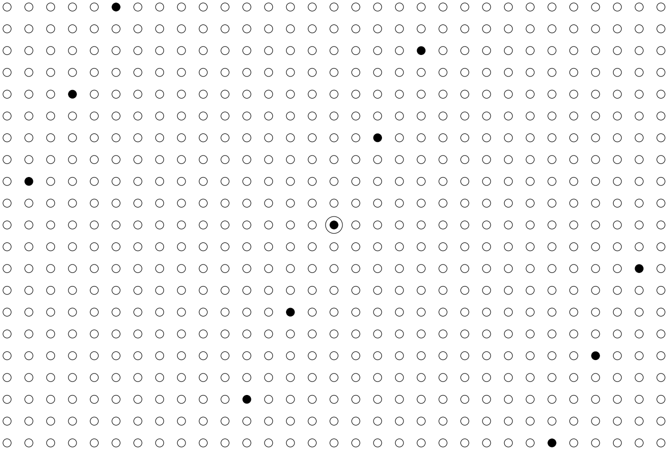
colouring of $\Gamma_1(R)$



colouring of $\Gamma_1(R^{-1})$ rotated by R



colouring of $\Gamma_1(R)$



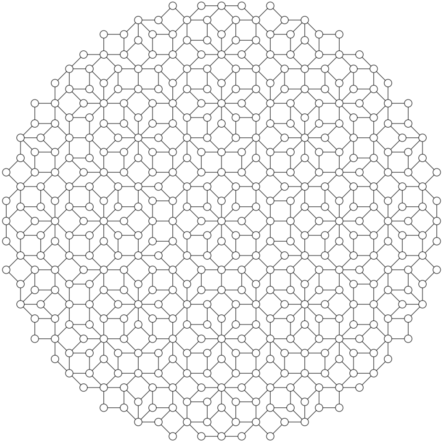
$\Gamma_2(R)$

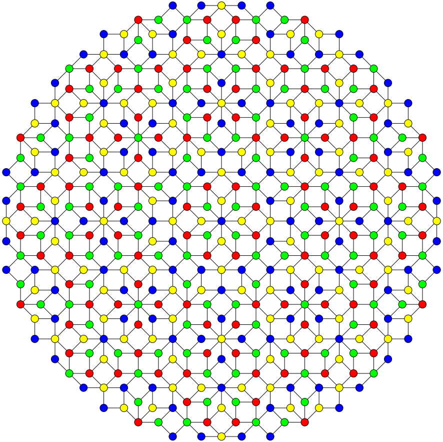
In our example:

$\Sigma_1(R) = 5$, $m = t = s = 6$, and $u = v = 2$

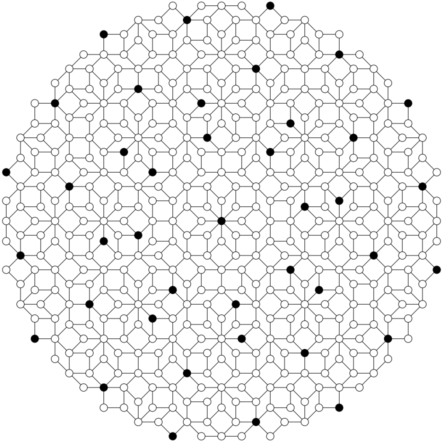
$\Rightarrow \Sigma_2(R) = 10$.

Example : Ammann-Beenker tiling



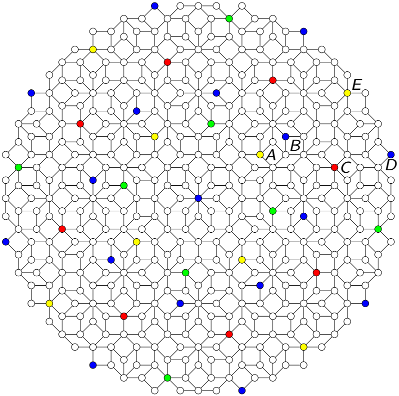


coloring induced by a submodule M_2 of index 4

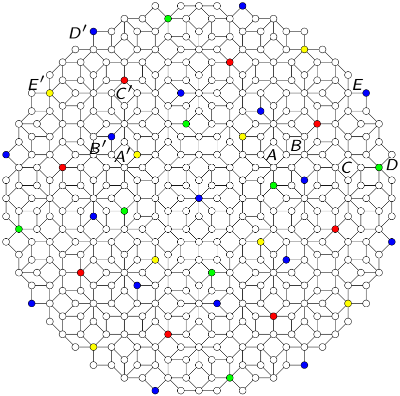


R the rotation about the center by $\theta = \arctan\left(-2\sqrt{2}\right) \approx 109.5^\circ$

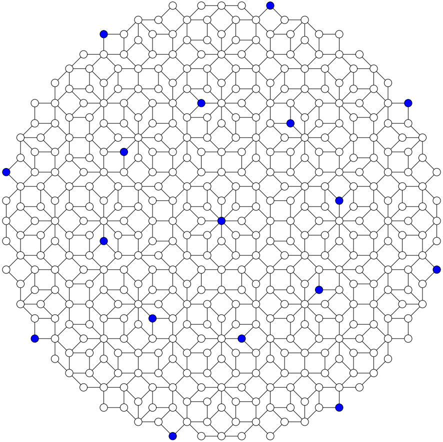
$(\Sigma_1(R) = 9$, acceptance factor $= 0.980572924\ldots)$



$T_2 \cap T_1(R^{-1})$



$T_2 \cap T_1(R)$



$T_2(R)$

colour coincidences

R is a colour coincidence

$\iff$

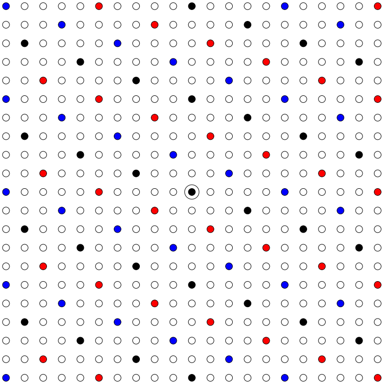
colouring of $\Gamma_1(R)$ is a rotated copy of the colouring of $\Gamma_1(R^{-1})$

(up to colour permutations)

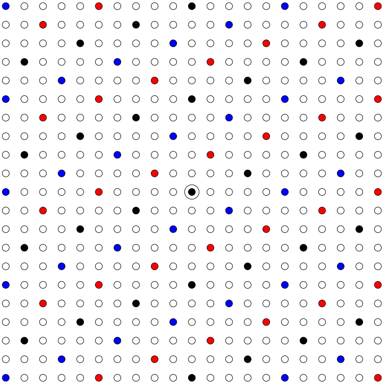
$\iff$

R leaves colour c_1 fixed

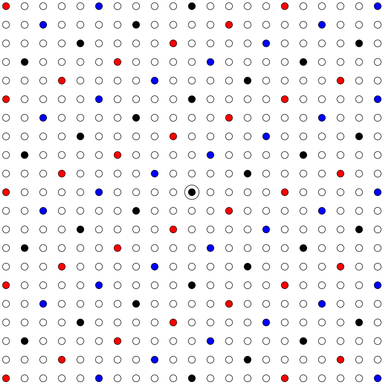
$\implies \Sigma_2 \mid \Sigma_1$



colouring of $\Gamma_1(R^{-1})$



colouring of $\Gamma_1(R)$



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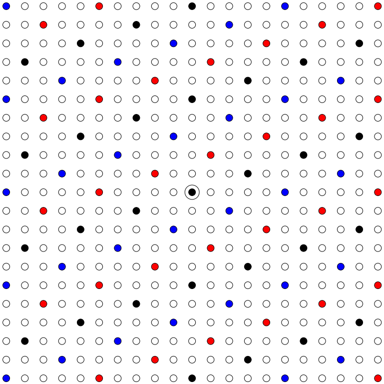

colouring of $\Gamma_1(R)$

Problem: Do colour coincidences form a group?

- R colour coincidence $\Longleftrightarrow$ R^{-1} colour coincidence
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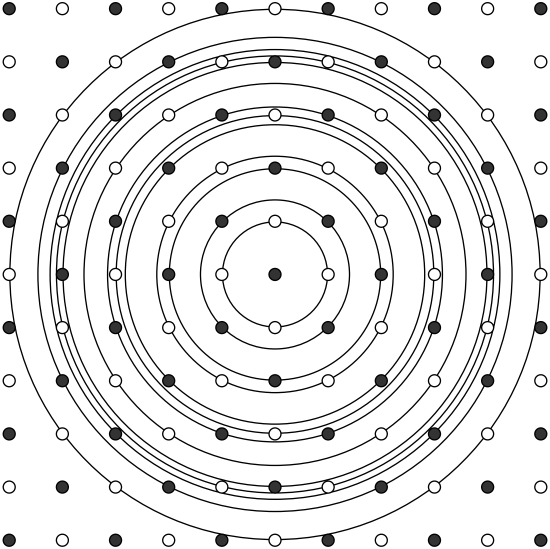

Theorem

different colours on different shells $\Longrightarrow$ *all coincidence rotations are colour coincidences*

Examples:

- primitive, body-centered, face-centered cubic lattices in dim=3
- primitive, body-centered, face-centered hypercubic lattices in $d \neq 0 \pmod 4$

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Thank you!