DEFINITION. An object is symmetrical if, after rotation, reflection, or some other rigid motion, it coincides with itself.

DEFINITION. A rigid motion is one that preseves the distances between any two points of space. A rigid motion is also called a

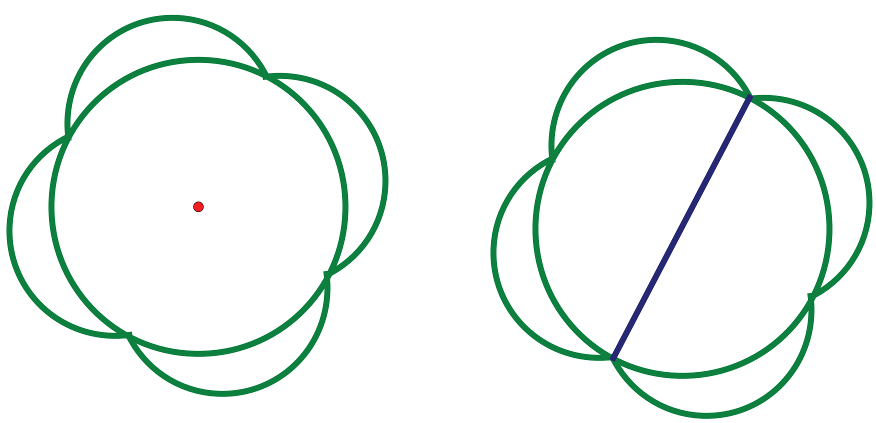
symmetry operation or an isometry.

isometry = iso + metric = same measure

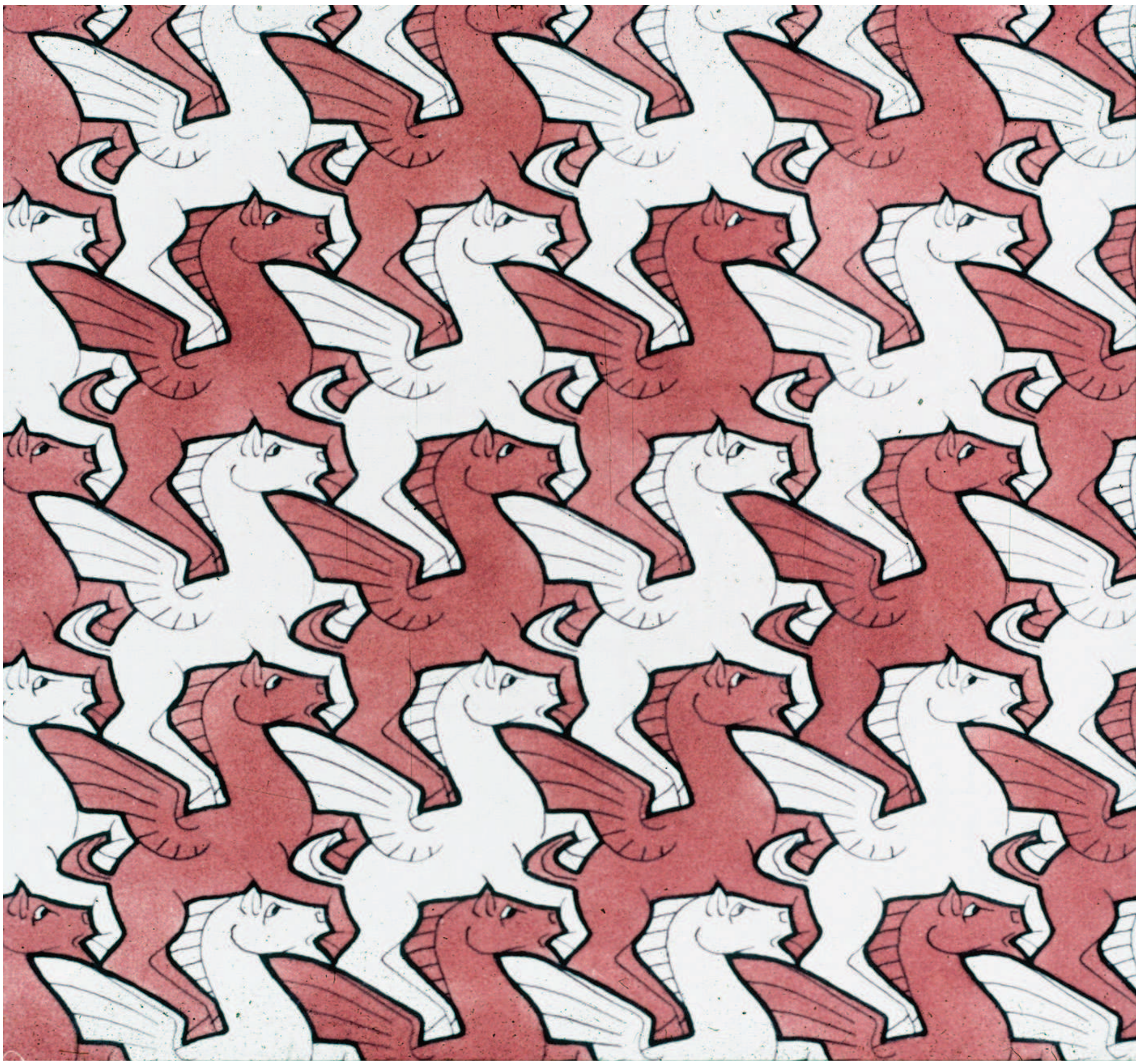
DEFINITION. The points of an object that are left fixed by a symmetry operation constitute a symmetry element of that object.

For example, in two dimensions these include:

OPERATION	NOTATION	ELEMENT
rotation	r	point
reflection	m	line



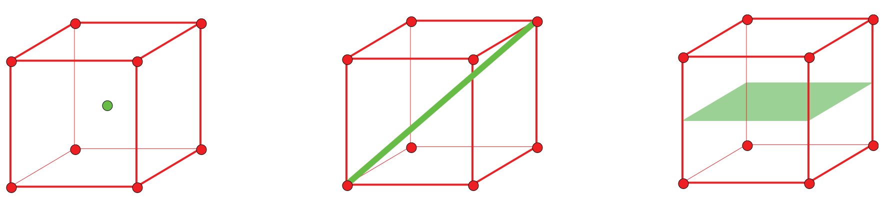
Another symmetry operation in the plane is translation t, a parallel shift. Translations have no fixed points.



But we'll leave translation for another lecture.

This morning, we consider only symmetry operations with at least one fixed point. In 3D these include

OPERATION	NOTATION	ELEMENT
inversion	i	point
rotation	r	line
reflection	m	plane



We can illustrate the effect of a symmetry operation on an object by marking any point on the object and its image under that action.

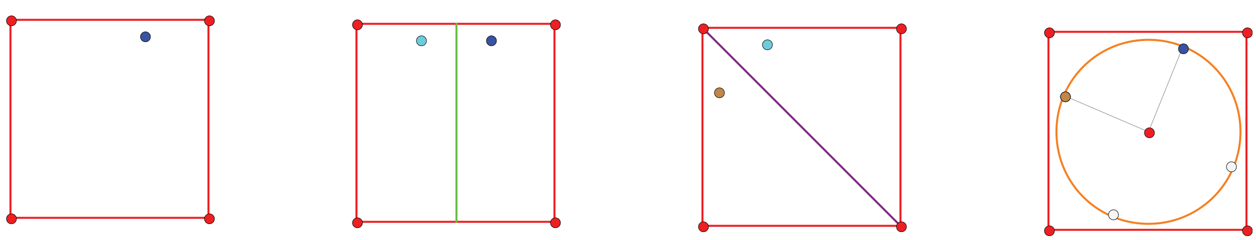
If two symmetry operations have the same effect on all points, they are considered to be the same.

For example, for an object IN THE PLANE (but not in 3D), 180 deg rotation and inversion in the center are the same.

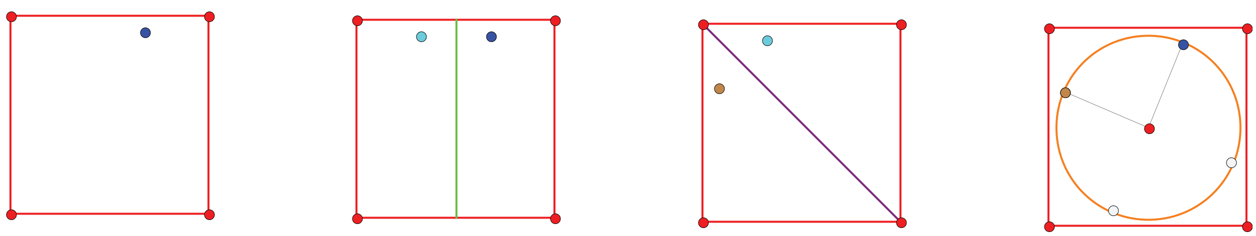
One symmetry operation followed by another is another symmetry operation. "Followed by" is a kind of multiplication or product $\circ$. For the square below,

reflection in $m1$ $\circ$ reflection in $m2 = r$

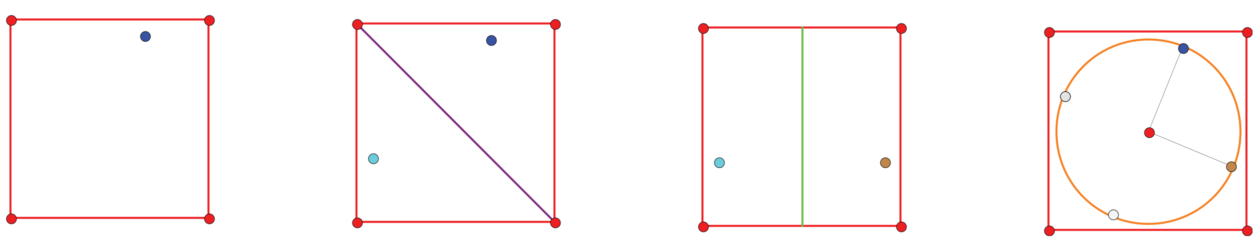
where r is 45 deg rotation about the center of the square.



But unlike ordinary number multiplication, for which $a \times b = b \times a$, the product $\circ$ does not always commute. Compare:



with



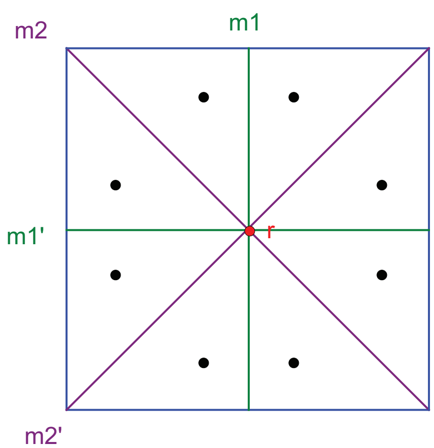
DEFINITION. The symmetry operation that fixes all points is called the identity operation. We denote the identity by the letter e.

Examples:
$m1 \circ m1 = e$
$m2 \circ m2 = e$
$r \circ r \circ r \circ r = r^4 = e$

DEFINITION. The number of times a symmetry operation must be repeated to "return" to the identity is its order.

The orders of $m1$ and $m2$ are 2; the order of r is 4.

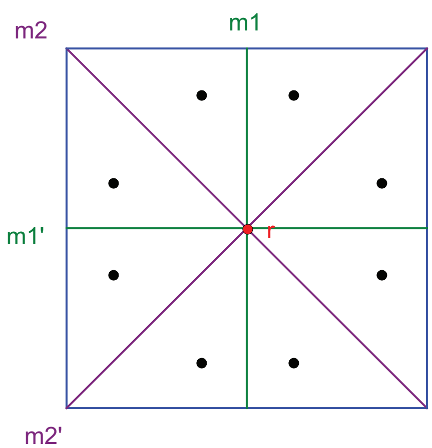
Fill in the multiplication table for the symmetry operations of the square.



do this operation first

	e	r	r2	r3	m1	m1'	m2	m2'
e	e							
r				e				
r2			e					
r3		e						
m1					e			
m1'						e		
m2							e	
m2'								e

Your table should be identical with this one:



do this operation first

	e	r	r2	r3	m1	m1'	m2	m2'
e	e	r	r2	r3	m1	m1'	m2	m2'
r	r	r2	r3	e	m2	m2'	m1'	m1
r2	r2	r3	e	r	m1'	m1	m2'	m2
r3	r3	e	r	r2	m2'	m2	m1	m1'
m1	m1	m2'	m1'	m2	e	r2	r3	r
m1'	m1'	m2	m1	m2'	r2	e	r	r3
m2	m2	m1	m2'	m1'	r	r3	e	r2
m2'	m2'	m1'	m2	m1	r3	r	r2	e

DEFINITION. The symmmetry operations of an object constitute its symmetry group.

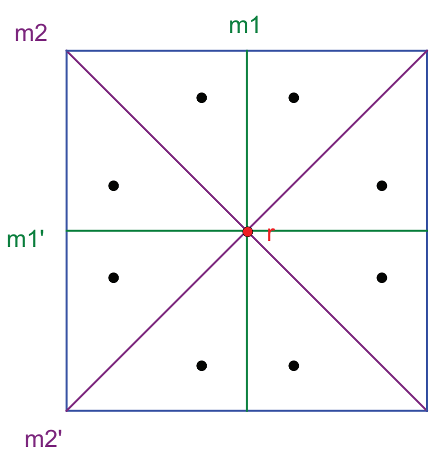
DEFINITION. A group is a set $G = \{e, g_1, g_2, g_3 \ldots\}$ together with a product $\circ$, such that

i) G is "closed under $\circ$": if g_1 and g_2 are any two members of G then so are $g_1 \circ g_2$ and $g_2 \circ g_1$;
ii) G contains an identity e: for any g in G, $e \circ g = g \circ e = g$;
iii) $\circ$ is associative: $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$;
iv) Each g in G has an inverse g^{-1} that is also in G: $g \circ g^{-1} = g^{-1} \circ g = e$.

Symmetry groups are a fundamental concept of mathematical crystallography.

If every element of G can can be written as a product of elements of some subset $\{g_1, \ldots, g_k\}$, then this subset generates G.

Find a set of two generators for the symmetry group of the square.

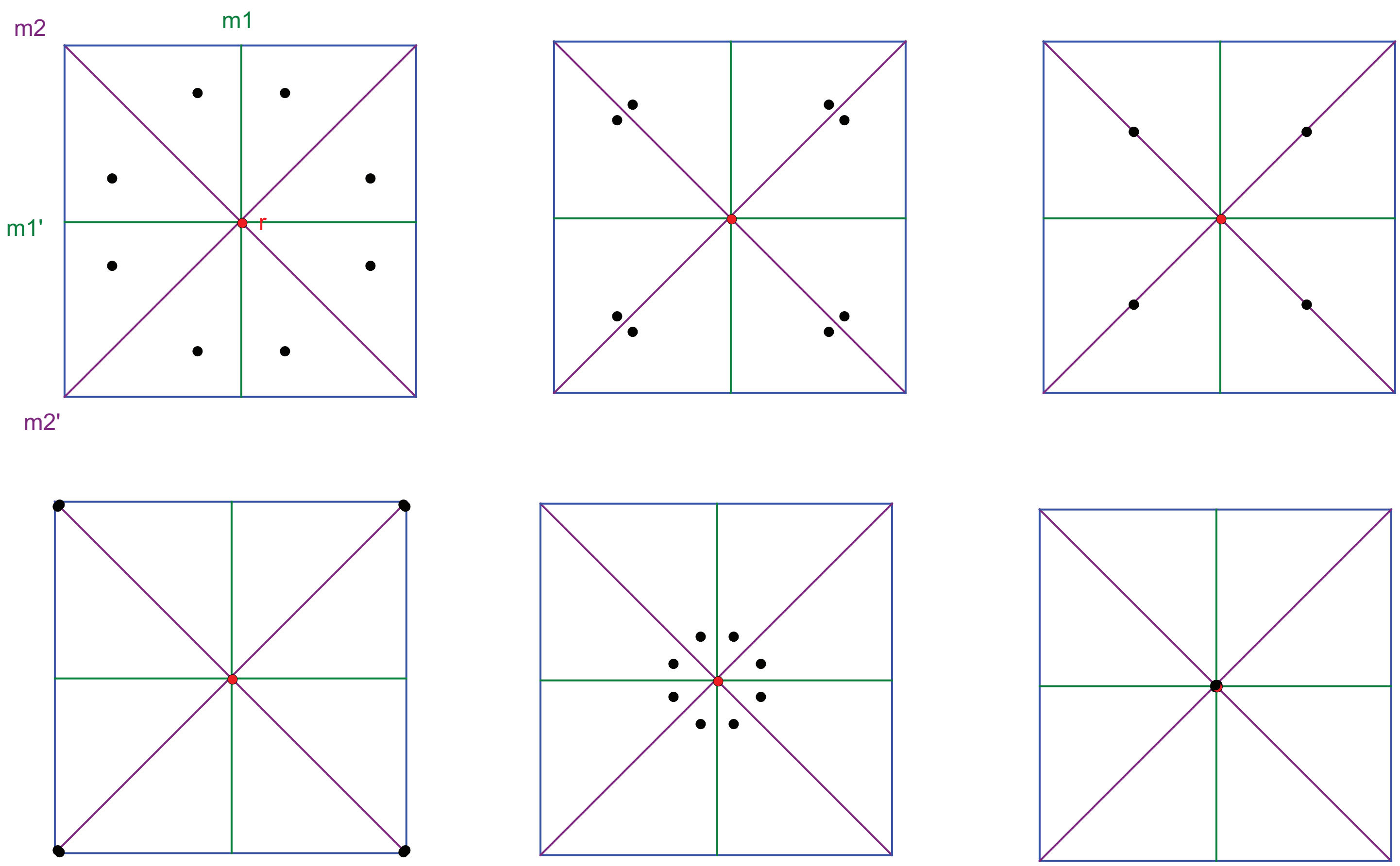


do this operation first

	e	r	r2	r3	m1	m1'	m2	m2'
e	e	r	r2	r3	m1	m1'	m2	m2'
r	r	r2	r3	e	m2	m2'	m1'	m1
r2	r2	r3	e	r	m1'	m1	m2'	m2
r3	r3	e	r	r2	m2'	m2	m1	m1'
m1	m1	m2'	m1'	m2	e	r2	r3	r
m1'	m1'	m2	m1	m2'	r2	e	r	r3
m2	m2	m1	m2'	m1'	r	r3	e	r2
m2'	m2'	m1'	m2	m1	r3	r	r2	e

Find another set of two generators for this group.

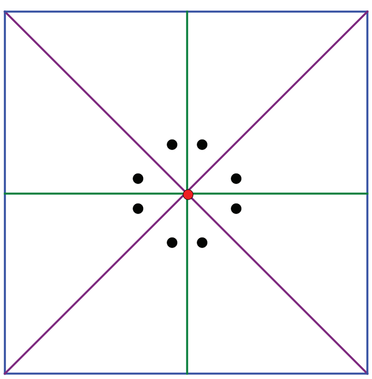
DEFINITION. The path of a point on an object under the action of its symmetry group G is an orbit of G. Here the orbits of some points of the square.



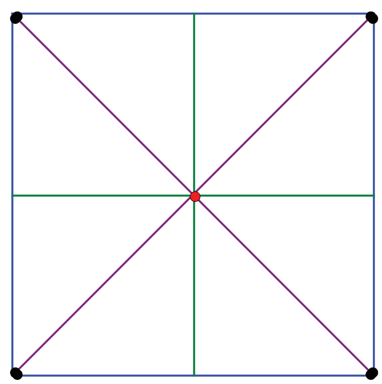
Since a point lying on a symmetry element is kept in place by the corresponding symmetry operation, the number of points in the orbit is correspondingly reduced.

DEFINITION. The set S_x of operations that keep a point S fixed is its site symmetry group.

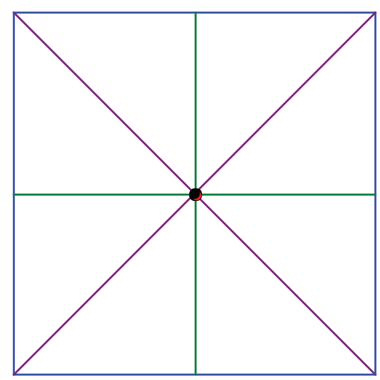
This terminology is appropriate because the operations that fix a point satisfy the group axioms.



S = {e}
|S| = 1



S = m2 or m2'
|S| = 2



S = G
|S| = 8

A little algebra shows that the number of points in the orbit of x is k, then $k \times |S| = |G|$ where $|S|$ and $|G|$ denote the corresponding number of symmetry operations, including e.