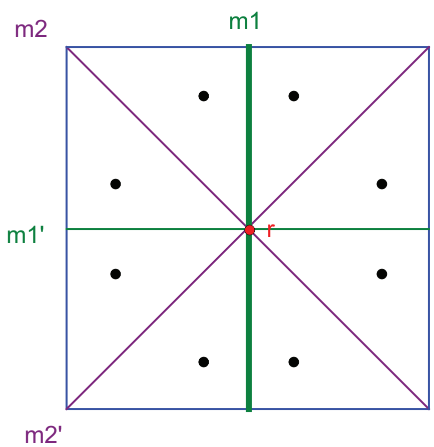
Definition. A *subgroup* of a symmetry group G is a subset H of symmetry operations that is itself a group.

To decide whether a subset is a subgroup, we have to check that
H contains the identity element e
H is closed under the operation $\circ$.

Examples:

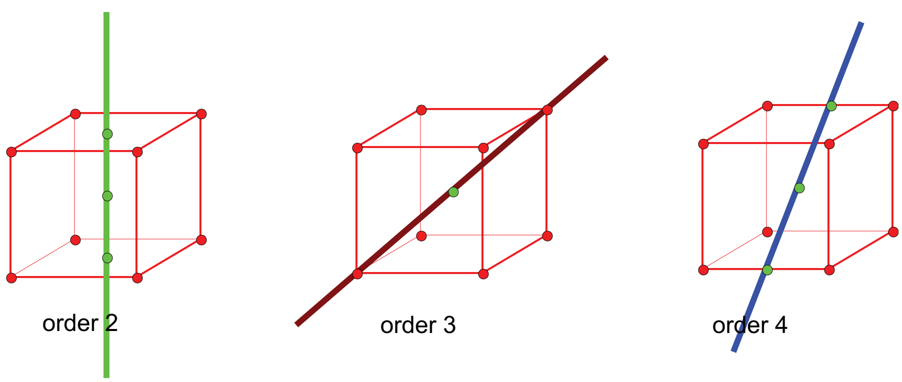
If G contains a reflection m, then $H = \{e, m\}$ is always a subgroup, since $m \circ m = e$.

The subset $\{e, r, m_1\}$ of the group of symmetries of the square is not a subgroup.



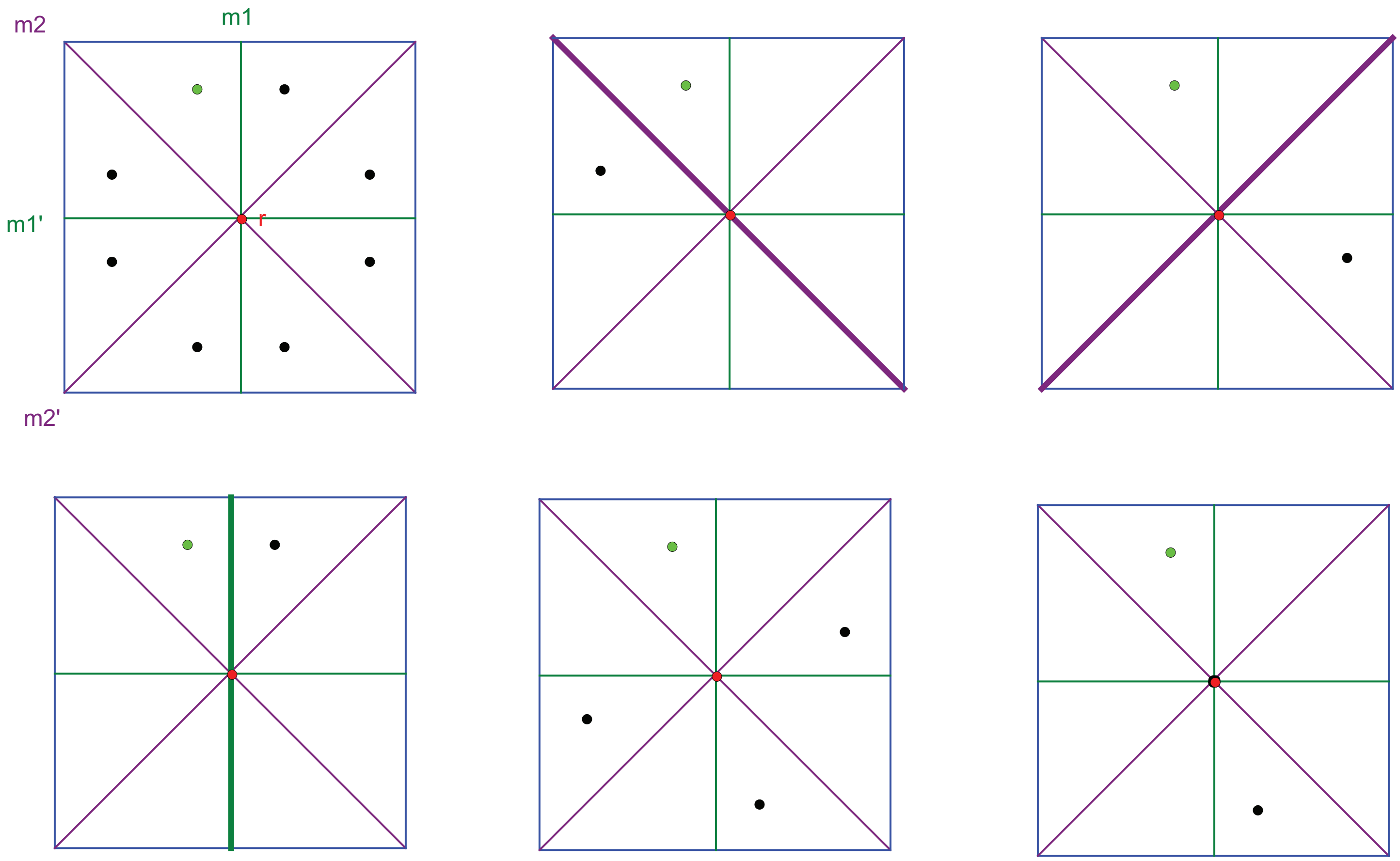
More examples:

The rotations about any axis of the cube; what is the size of the subgroup in each case?



And the rotations about all the axes of the cube; what is the size of this subgroup?

If O is an orbit of G whose site symmetry groups are just the identity, then the orbit of a subgroup H is a subset of O.



If H is a subgroup of G and g is not in H, then the operations of the form $g^{-1}Hg$ form a subgroup too, since

$g^{-1} \circ e \circ g = e$ is in $g^{-1}Hg$ and

$$(g^{-1} \circ h_1 \circ g) \circ (g^{-1} \circ h_2 \circ g) =$$

$$(g^{-1} \circ h_1) \circ (g \circ g^{-1}) \circ (h_2 \circ g) =$$

$$g^{-1} \circ (h_1 \circ h_2) \circ g.$$

Definition. The subgroups H and $g^{-1}Hg$ are conjugate.

Let $H = \{e, h_1, h_2, \ldots, h_k\}$ be a subgroup of G and let g be any element of G that is not in H.

Definition. The set $gH = \{g, g \circ h_1, g \circ h_2, \ldots, g \circ h_k\}$ is a left coset of H.

For example, let G be the symmetry group of the square and H the subgroup $\{e, m_1\}$. Then $m_2H = \{m_2, m_2 \circ m_1\}$.

Consulting the group table, we find that $m_2 \circ m_1 = r$. Thus $m_2H = \{m_2, r\}$.

Continuing, we have $r^2H = \{r^2, m_1'\}$ and $r^3 = \{r^3, m_2'\}$.

More about cosets.

Right cosets Hg are defined similarly.

Each coset, left or right, of any subgroup H consists of the same number of operations of G.

No two left cosets – and no two right cosets – have an operation in common.

Every operation of G belongs to exactly one left coset of H (and to exactly one right coset).

Thus the left (and the right) cosets of a subgroup H partition the group G into disjoint sets.

Definition. The index of a subgroup H of G is the number of cosets of H in G.

Lagrange's Theorem (for finite groups)

The order of a subgroup divides the order of the group.

Definition. If the left cosets and right cosets of a subgroup H are identical – if $gH = Hg$ – then H is a normal subgroup of G.

Every subgroup of index 2 is normal, since it has only one coset.

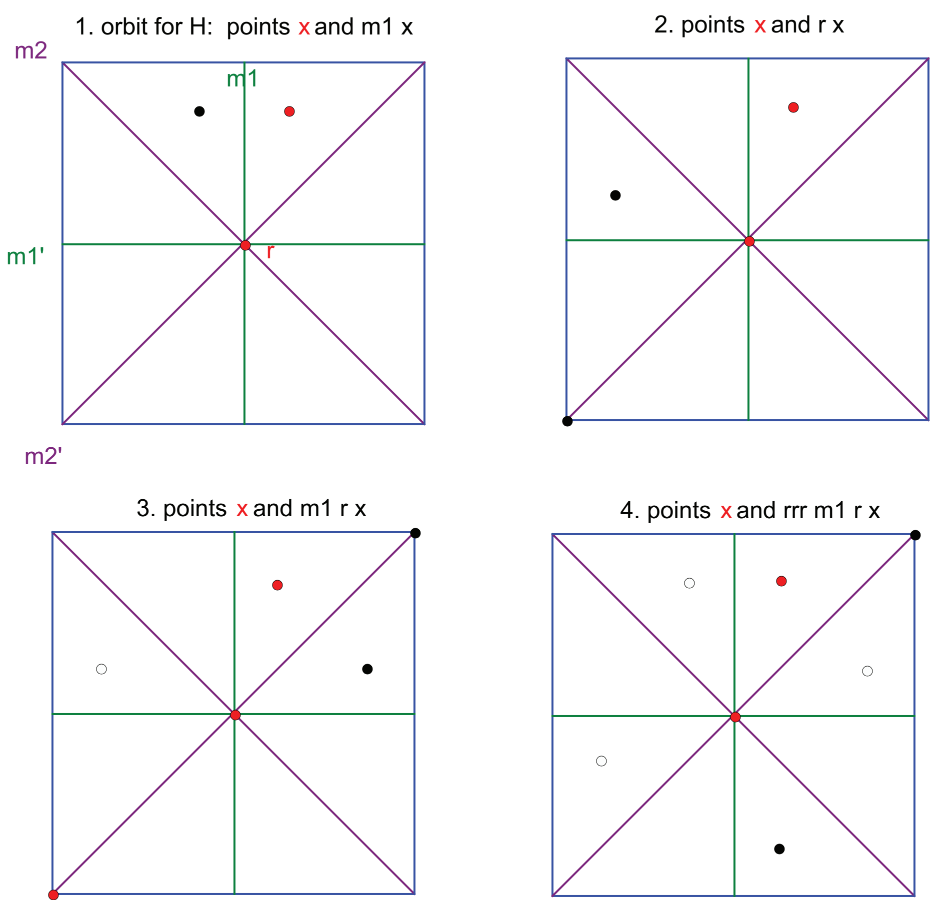
If H is normal, then $gh_i = h_j g$ for some h_j in H which may be different from h_i. Thus normality does not imply commutativity.

Another way of writing $gH = Hg$ is $g^{-1}Hg = H$. So a normal subgroup H is conjugate to itself.

How to know a non-normal subgroup when you see one.

Consider the square again, and the subgroup $H = \{e, m_1\}$. For g, choose any operation not in H, say the 4-fold rotation r.

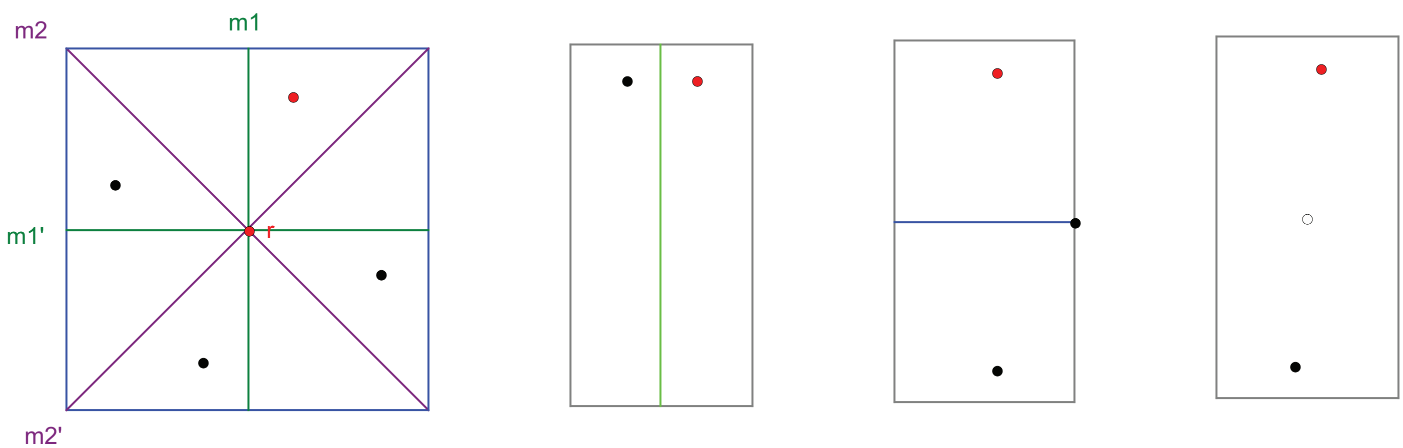
Below, an orbit for H is shown at the left, and an orbit for $g^{-1}Hg$ is shown at the right.



A conjugate $g^{-1}Hg$ of H acts the same as H but somewhere else!.

For a normal subgroup, there is nowhere else. That is, normal subgroups play unique roles in G.

Some normal subgroups:



Quotient Groups

The cosets $eH, g_1H, g_2H, \ldots, g_kH$ of a normal subgroup H form a group:

$$g_1H \circ g_2H = (g_1 \circ g_2)H$$

This group, denoted G/H, is called a "factor group" or the "quotient of G by H".

Which suggests that G is, in some sense, a product of H and G/H.

Let the cosets of H in G be $eH, g_1H,\ g_2H, \ldots, g_kH$ and let S be the set of "coset representatives",

$S = \{e, g_1,\ g_2,\ \ldots,\ g_k\}$.

We can distinguish three cases:

1) If S, like H, is a normal subgroup of G, then G is a direct product of H and S.

2) If S is a subgroup of G but not a normal subgroup, then G is a semi-direct product of H and S.

3) If S is not a subgroup of G, then G is some other sort of "product" of H and S.

And that's what the space groups are all about.