

Definition. A **lattice** Λ in n -dimensional Euclidean space is a set of points with two properties:

i) there is a positive number r such for any points x and y of Λ , the distance $|x - y|$ is $\geq r$, and

ii) if x and y are points of Λ then the translation mapping x to y maps all of Λ onto itself.

An example in E^3 :



Example: the cubic lattice in E^5 .

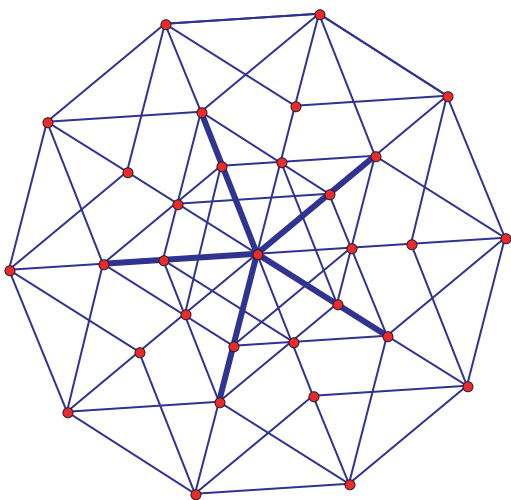
The points of this lattice have integer coordinates in the standard Cartesian coordinate system. For example, a unit cube with one vertex at the origin and edges parallel to the coordinate axes has vertices 10000, 01000, ... 11111.

We denote points by their coordinates: in 5D space, 11111; in 3D space 234, etc, without brackets.

The same coordinates in brackets: [11111], [234], etc indicate vectors.

In E^5 , the minimum distance between any two points of this lattice is 1. To see why the second condition holds, let $x = abcde$ and $y = fghij$ be two lattice points (The letters stand for arbitrary integers.) Then the translation carrying x to y is the vector $y - x = [f-a \ g-b \ h-c \ i-d \ j-e]$. If z is a third lattice point, say $z = klmno$, then this translation carries z to $z + y - x = [k+f-a \ . \ . \ . \ .]$.

Here is a projection of the unit five-dimensional cube, along the direction $[11111]$, onto a plane.



Vectors can be added.

$$[ab] + [cd] = [a + b \ c + d];$$

$$[a_1 a_2 \dots a_n] + [b_1 b_2 \dots b_n] = [a_1 + b_1 \ a_2 + b_2 \ \dots \ a_n + b_n].$$

Thus the sum of two vectors is a vector.

Addition is commutative.

Since $[a \ b] + [\bar{a} \ \bar{b}] = [0 \ 0]$, etc., every vector has an inverse.

With vector addition as the "product" and the zero vector $[0 \ 0 \ \dots \ 0]$ as the "identity", vectors form a group.

Vectors can be multiplied by any real numbers:
 $2[34] = [68]$; $\sqrt{2}[34] = [3\sqrt{2} \ 4\sqrt{2}]$.

Definition. A sum of the form $\alpha[a_1a_2\dots a_n] + \beta[b_1b_2\dots b_n]$, where α and β are any real numbers, is a **linear combination** of $[a_1a_2\dots a_n]$ and $[b_1b_2\dots b_n]$.

The numbers α and β are the coefficients of the linear combination. If α and β are integers the combination is said to be **integral**.

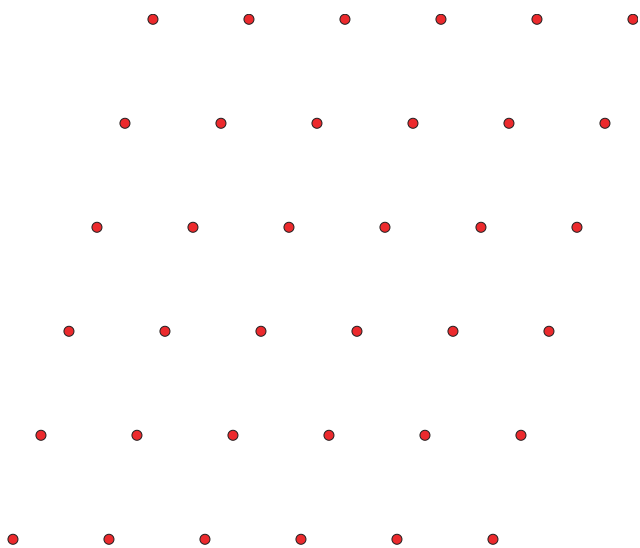
Definition. n vectors in n -dimensional space are said to be a **basis** for that space if every vector can be written a linear combination of those n vectors. Notice that the number of vectors in a basis equals the dimension.

$[100]$, $[010]$ and $[001]$ are a basis for 3-dimensional space because any vector $[\alpha \ \beta \ \gamma]$ can be written in the form $\alpha[100] + \beta[010] + \gamma[001]$.

On the other hand, $[100]$, $[010]$, and $[110]$ are not a basis because the third coordinate of any linear combination of these three vectors is zero.

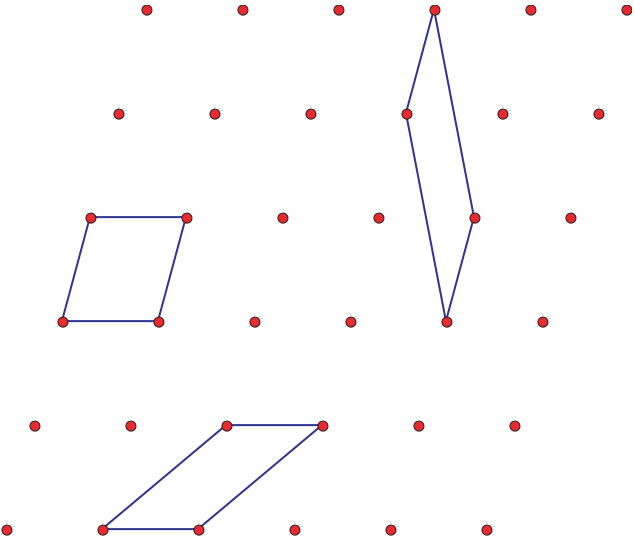
Definition The set of all **integral** linear combinations of a basis is a **vector lattice**.

Definition. An orbit of a vector lattice is a point lattice.



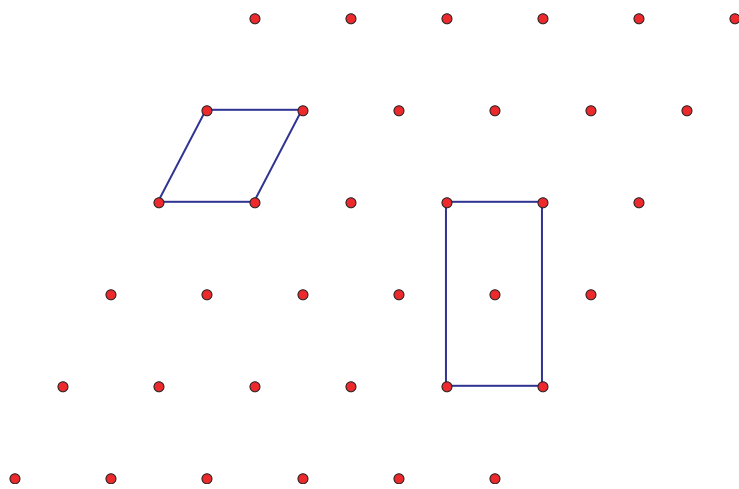
The basis vectors of a lattice are the edges of a parallelepiped whose vertices are lattice points, and which contains no other lattice points.

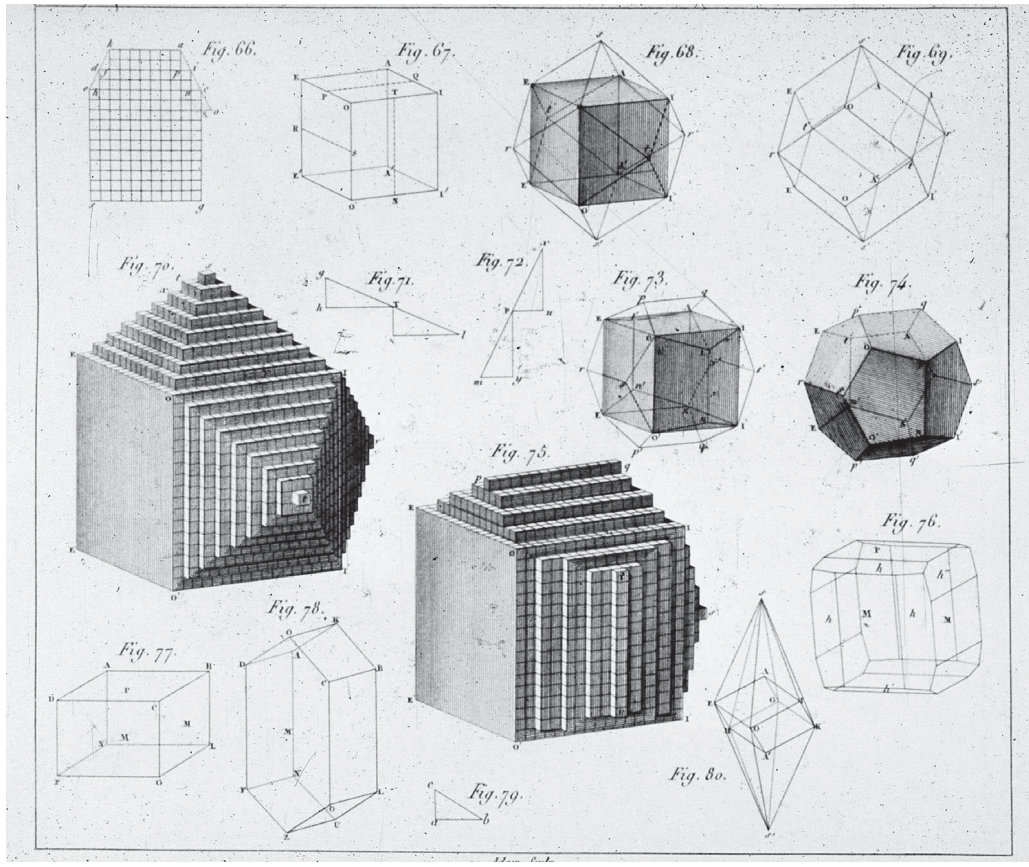
Conversely, the edges of **any** such parallelepiped are a basis for the lattice.



A parallelepiped with lattice points only at its vertices is called, in crystallography, a **primitive unit cell**.

Crystallographers also work with **centered** unit cells, when they better convey the symmetry of the lattice.





Crystallographers have studied cells, lattices, and the relations between them for nearly 200 years and research is still going on.

The **Voronoi** cell of a lattice point x is the set of points in space closer to x than to any other lattice point, together with the boundary of this set. Its symmetry group is the site symmetry group of x . The Voronoi cells of any two points of the same lattice are identical, and together they fill space. But Voronoi cells can be complicated geometrically; except in a few special cases, they are not parallelepipeds.

