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Commission on Mathematical and Theoretical Crystallography

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International school on fundamental crystallography:

Introduction to International Tables for Crystallography, Vol. A: Space-group symmetry and Vol. A1Symmetry relations between space groups

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CRYSTALLOGRAPHIC POINT GROUPS

(short review)

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I. Crystallographic symmetry operations

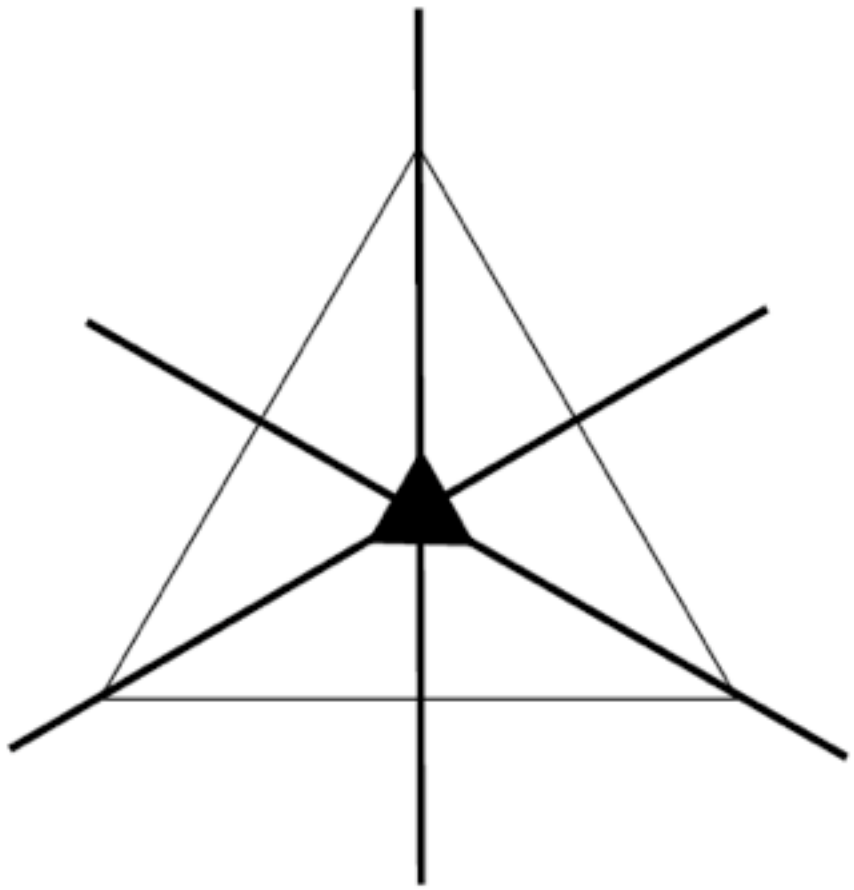
Symmetry operations of an object

The symmetry operations are *isometries, i.e.* they are special kind of *mappings* between an object and its image that leave all distances and angles invariant.

The isometries which map the object onto itself are called *symmetry operations of this object*. The *symmetry* of the object is the set of all its symmetry operations.

Crystallographic symmetry operations

If the object is a crystal pattern, representing a real crystal, its symmetry operations are called *crystallographic symmetry operations*.

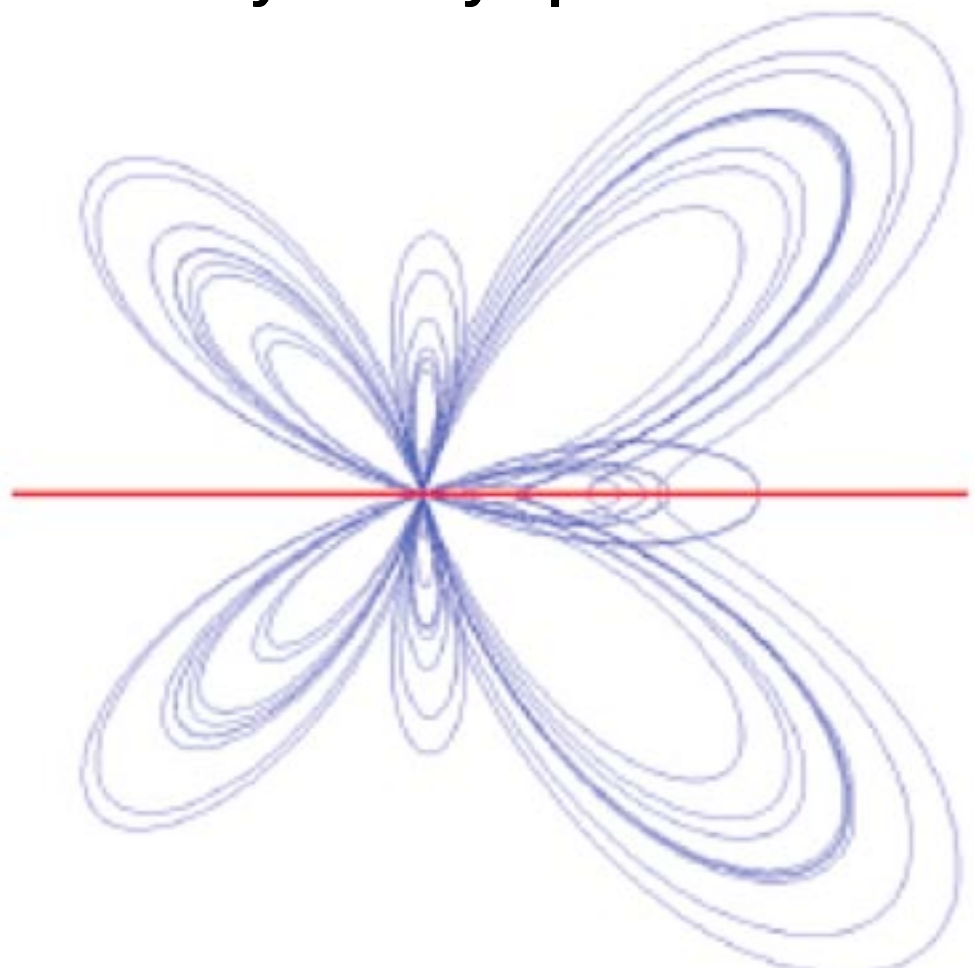


The equilateral triangle allows six symmetry operations: rotations by 120 and 240 around its centre, reflections through the three thick lines intersecting the centre, and the identity operation.

GROUP THEORY
(few basic facts)

Crystallographic symmetry operations in the plane

Mirror symmetry operation



drawing: M.M. Julian
Foundations of Crystallography
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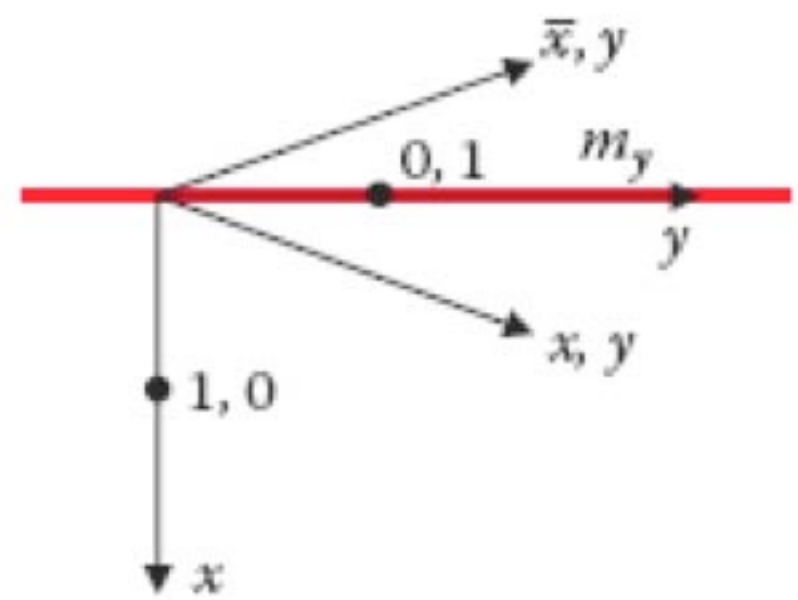
Mirror line m_y at 0,y



Matrix representation

$$m_y \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

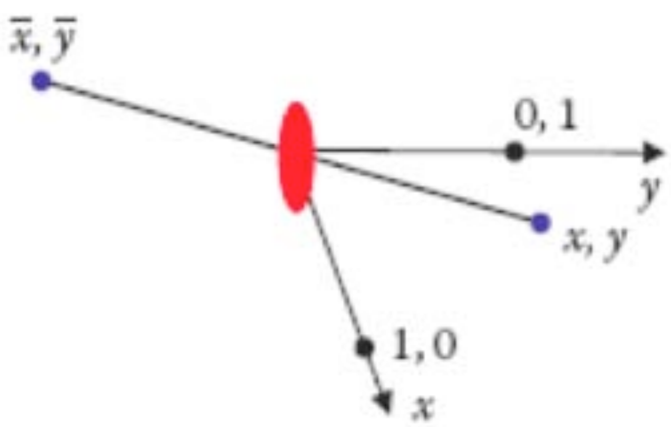
$$\det \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} = ? \qquad \mathrm{tr} \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} = ?$$

Fixed points

$$m_y \begin{pmatrix} x_f \\ y_f \end{pmatrix} = \begin{pmatrix} x_f \\ y_f \end{pmatrix}$$

Symmetry operations in the plane
Matrix representations

2-fold rotation

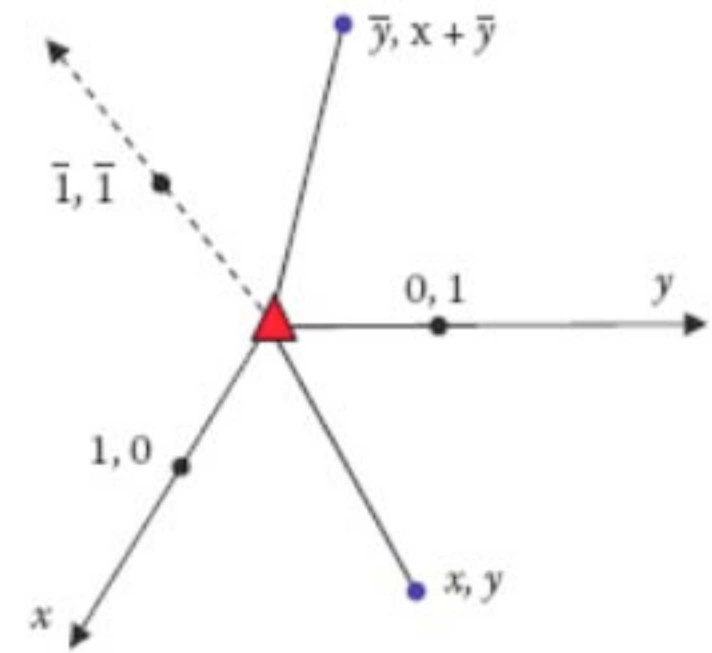


$$2_z \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -x \end{pmatrix} = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\det \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} = ?$$

$$\mathrm{tr} \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} = ?$$

3-fold rotation



$$3^+ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x-y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\det \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = ?$$

$$\mathrm{tr} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = ?$$

GROUP AXIOMS

1. CLOSURE

$$g_1 \circ g_2 = g_{12} \qquad g_1, g_2, g_{12} \in G$$

2. IDENTITY

$$g \circ e = e \circ g = g$$

3. INVERSE ELEMENT

$$g \circ g^{-1} = e$$

4. ASSOCIATIVITY

$$(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3) = g_1 \circ g_2 \circ g_3$$

Group properties and presentation

1. **Order of a group**

number of elements

2. **Multiplication table**

	E	A	B
E	E	A	B
A	A	B	E
B	B	E	A

3. **Group generators**

a set of elements such that each element of the group can be obtained as a product of the generators

4. How to define a group

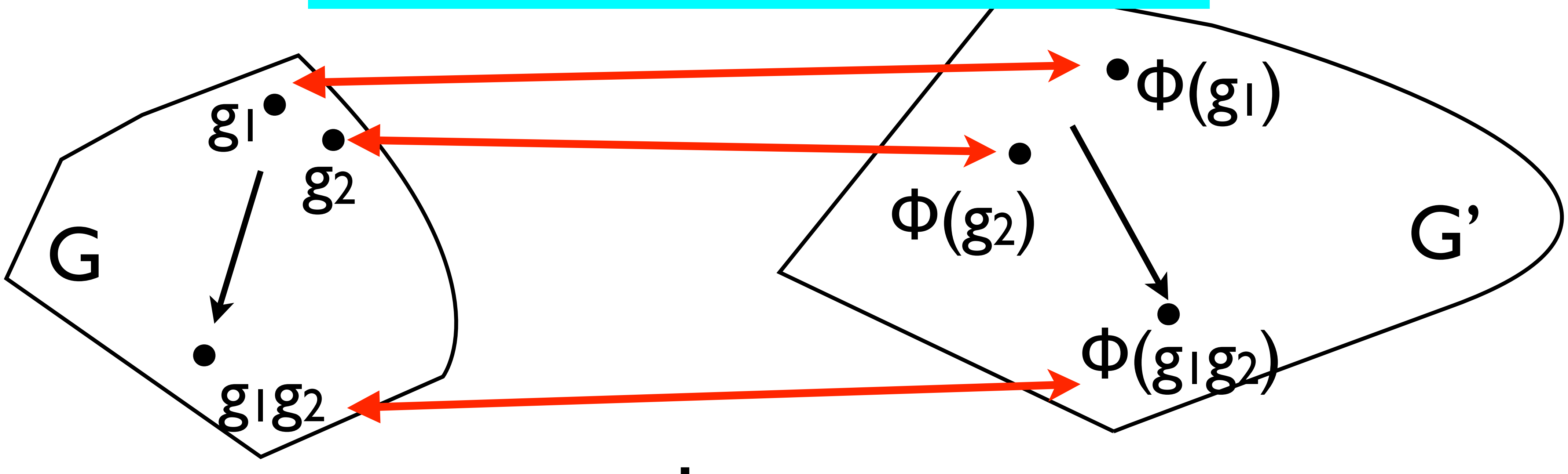
Multiplication table

	E	A	B
E	E	A	B
A	A	B	E
B	B	E	A

Group generators

a set of elements such that each element of the group can be obtained as a product of the generators

Isomorphic groups



$$\Phi(g)=g'$$

$$G=\{g\} \longleftrightarrow G'=\{g'\}$$

$$\Phi^{-1}(g')=g$$

$$\Phi: G \longrightarrow G' \quad \Phi^{-1}: G' \longrightarrow G$$

homomorphic condition $\Phi(g_1)\Phi(g_2)= \Phi(g_1g_2)$

-groups with the same multiplication table

Crystallographic Point Groups in 2D

Point group **2** = {1,2}

Motif with symmetry of **2**

drawing: M.M. Julian
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Where is the two-fold point?

-group axioms?

$$2 \times 2 = \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} \times \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

-order of **2**?

-multiplication table

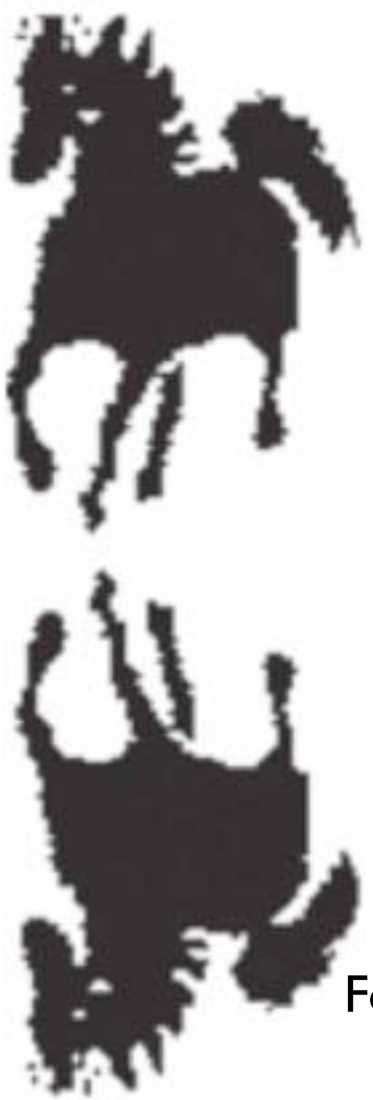
×	1	2
1	1	2
2	2	1

-generators of **2**?

Crystallographic Point Groups in 2D

Point group **m** = {1,m}

Motif with symmetry of **m**



Where is the mirror line?

drawing: M.M. Julian
Foundations of Crystallography

-group axioms?

$$m \times m = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix} \times \begin{bmatrix} -1 & \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

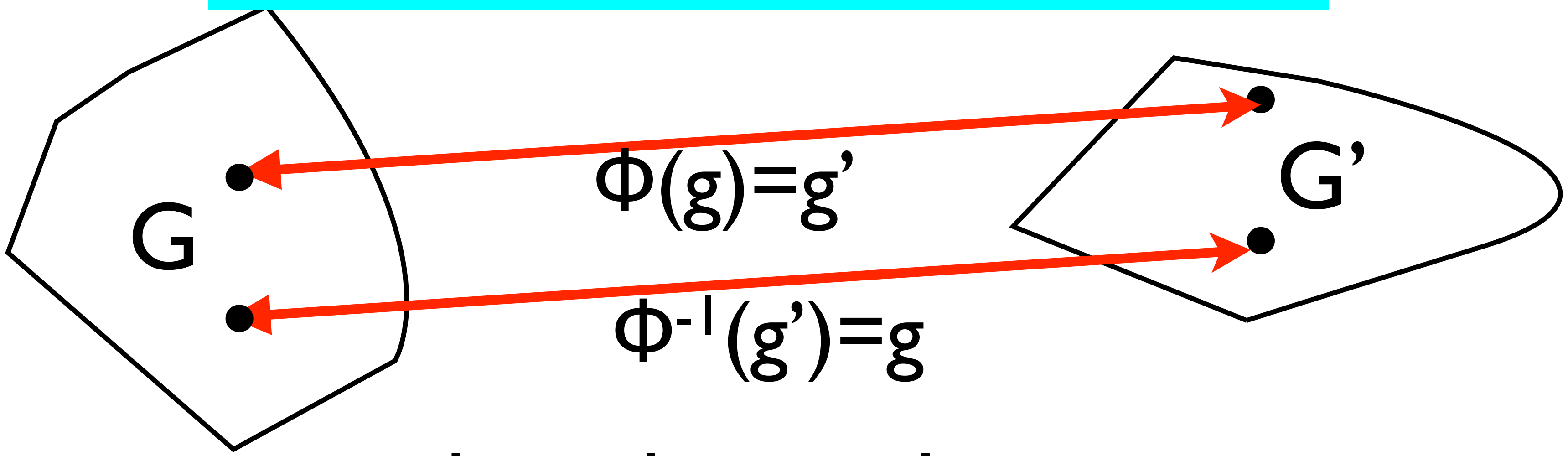
-order of **m**?

-multiplication table

$\times$	1	m_y
1	1	m_y
m_y	m_y	1

-generators of **m**?

Isomorphic groups



$$\Phi(g_1)\Phi(g_2)= \Phi(g_1\ g_2)$$

Point group **2** = {1,2}

×	1	2
1	1	2
2	2	1

Point group **m** = {1,m}

×	1	m_y
1	1	m_y
m_y	m_y	1

-groups with the same multiplication table

Crystallographic Point Groups in 2D

Point group **1** = {1}

Motif with symmetry of **1**

drawing: M.M. Julian
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-group axioms?

$$1 \times 1 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \times \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

-order of **1**?

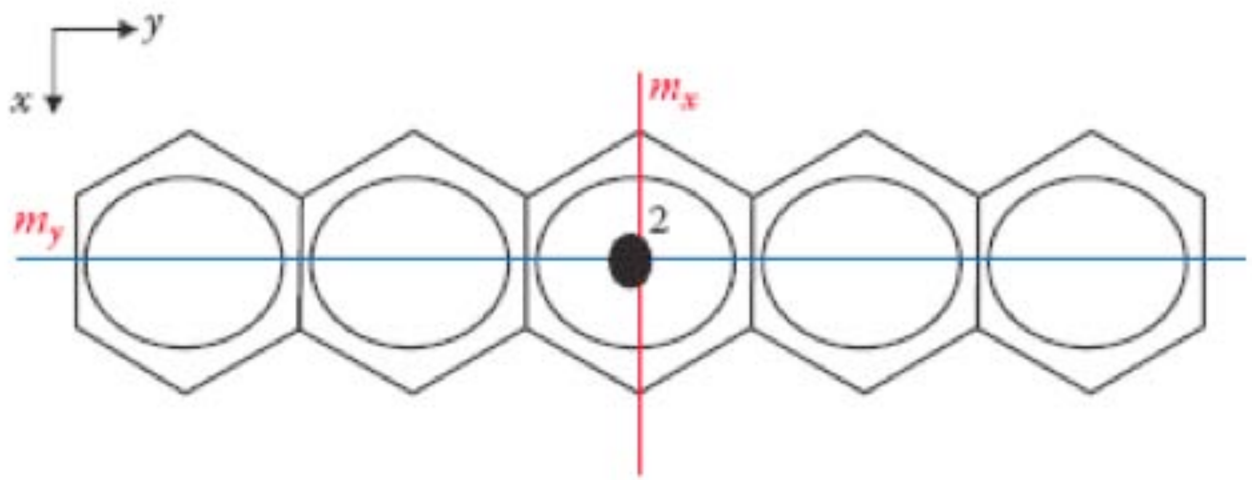
-multiplication table

×	1
1	1

-generators of **1**?

Problem 2.7

Consider the model of the molecule of the organic semiconductor pentacene ($C_{22}H_{14}$):



Determine:

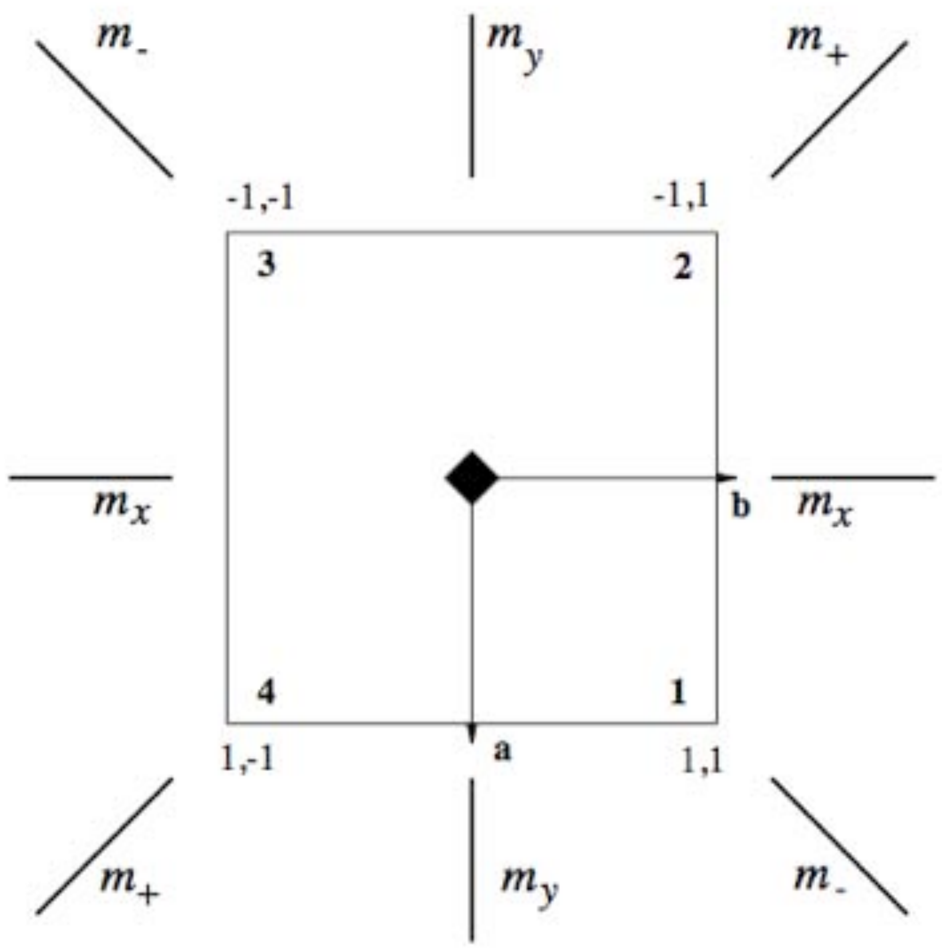
-symmetry operations:
matrix and (x,y) presentation

-generators

-multiplication table

Problem 2.8

Consider the symmetry group of the square. Determine:

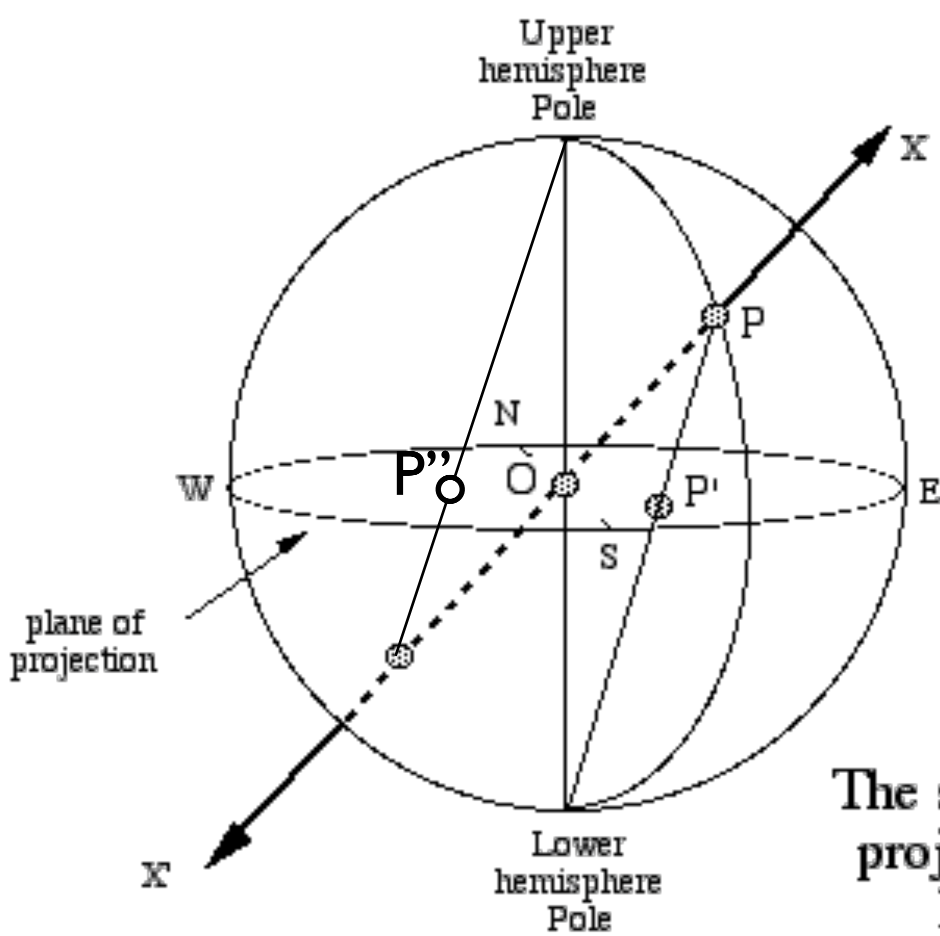


- -symmetry operations: matrix and (x,y) presentation
- -generators
- -multiplication table

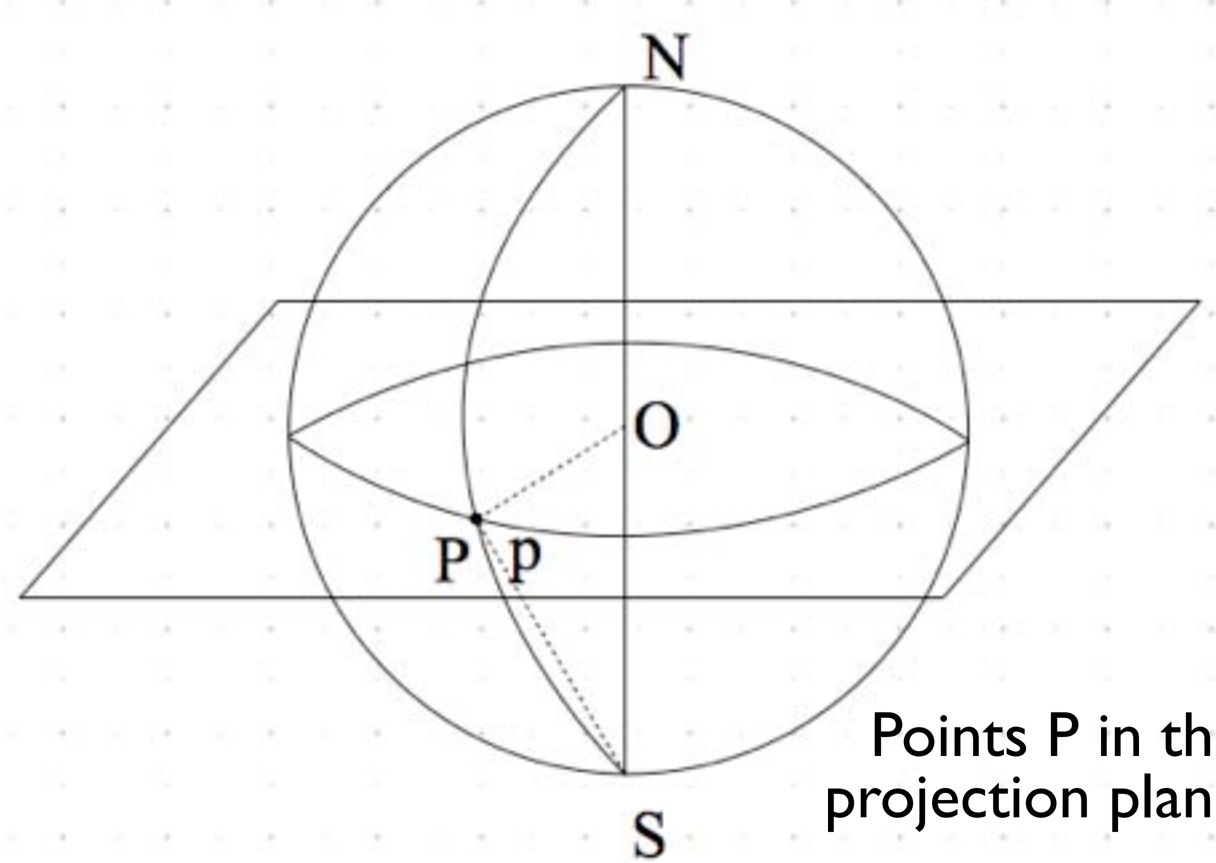
Visualization of Crystallographic Point Groups

- general position diagram
- symmetry elements diagram

Stereographic Projections



The stereographic projection of the line X-X'



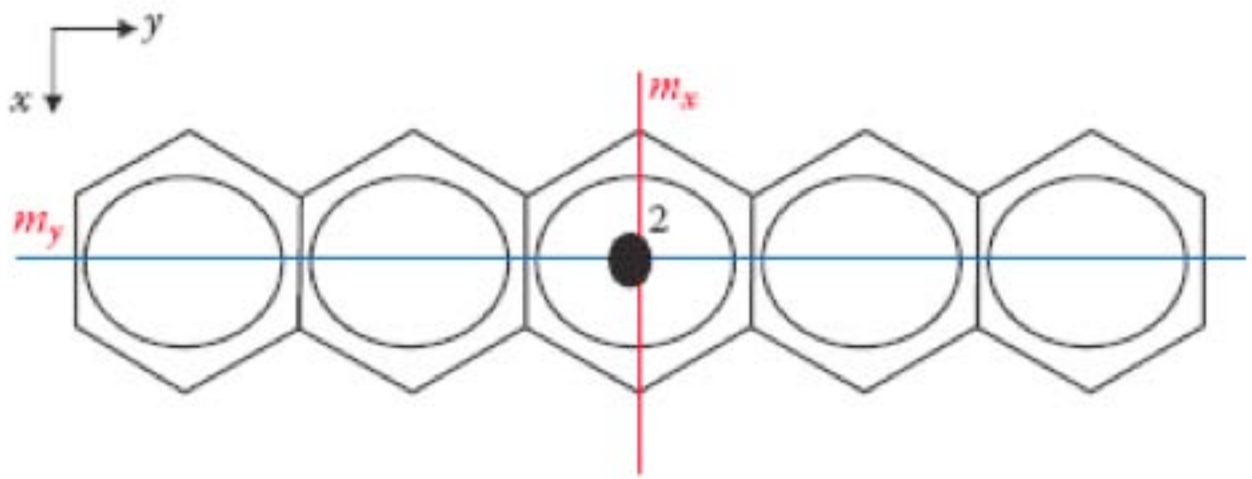
Points P in the projection plane

EXAMPLE

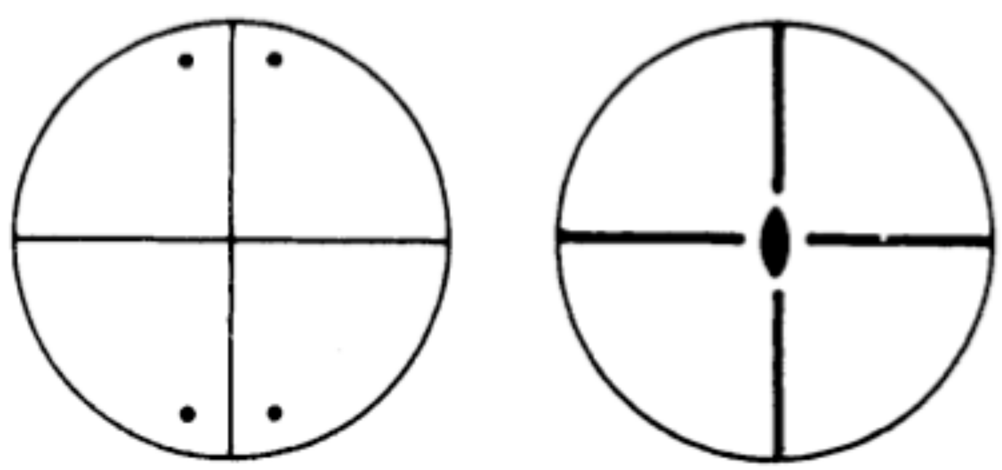
Stereographic Projections of ***mm2***

Point group **mm2** = {1,2_z,m_x,m_y}

Molecule of pentacene



Stereographic projections diagrams

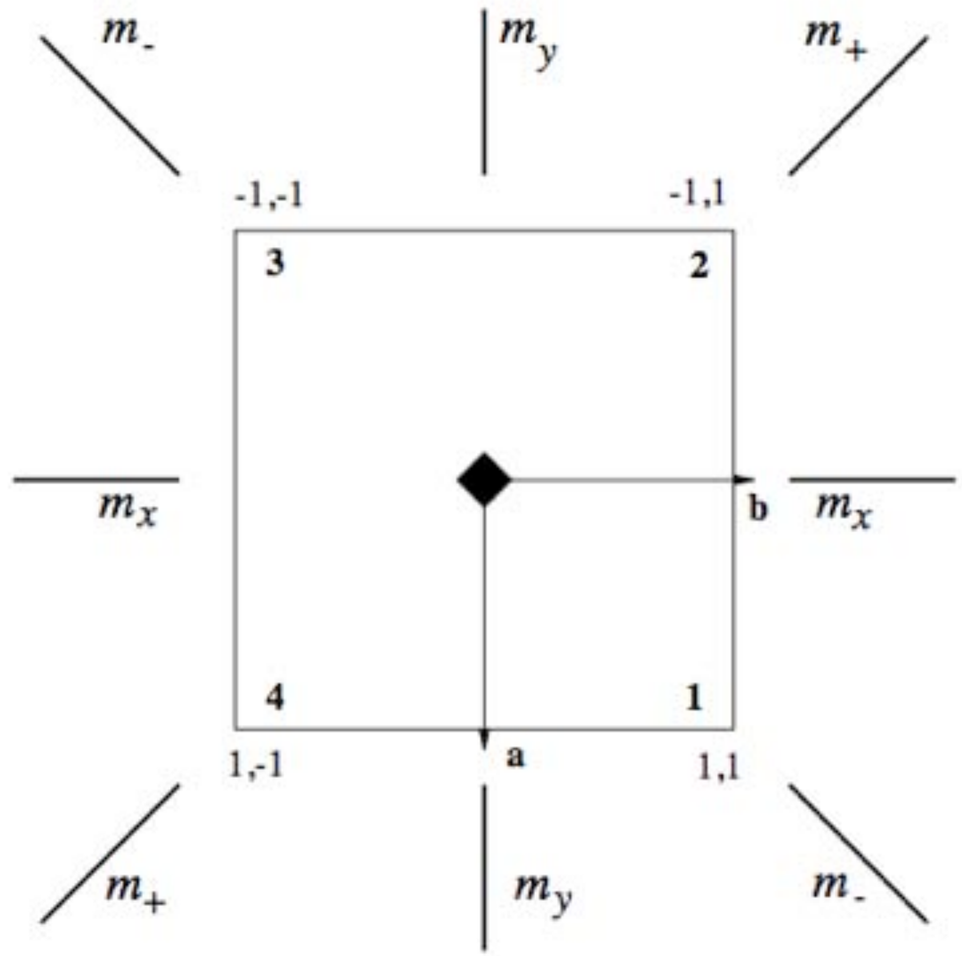


general position

symmetry elements

Problem 2.8 (cont.)

Stereographic Projections of ***4mm***



general position diagram

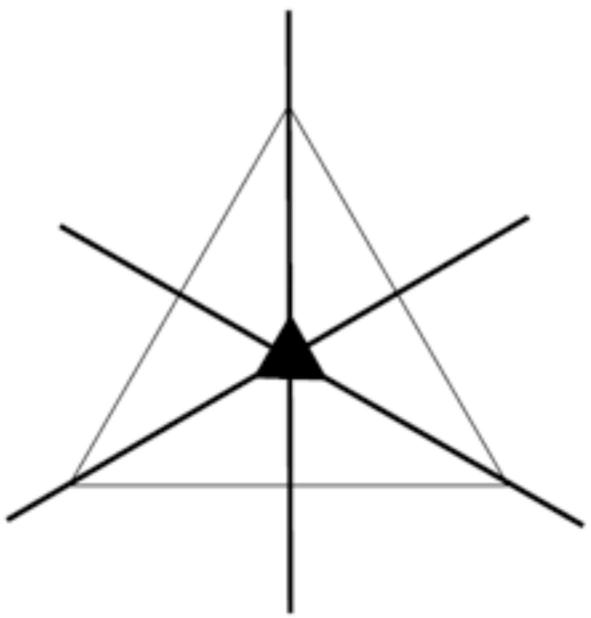
?

symmetry elements diagram

?

Problem 2.9 (additional)

Consider the symmetry group of the equilateral triangle. Determine:



-symmetry operations: matrix and (x,y) presentation

-general-position and symmetry-elements stereographic projection diagrams;

-generators

-multiplication table

Conjugate elements

Conjugate elements

$g_i \sim g_k$ if $\exists$ g: $g^{-1} g_i g = g_k$,
where $g, g_i, g_k, \in G$

Classes of conjugate elements

$L(g_i)=\{g_j |\ g^{-1} g_i g = g_j, g \in G\}$

Conjugation-properties

(i) $L(g_i) \cap L(g_j) = \{\varnothing\}$, if $g_i \notin L(g_j)$

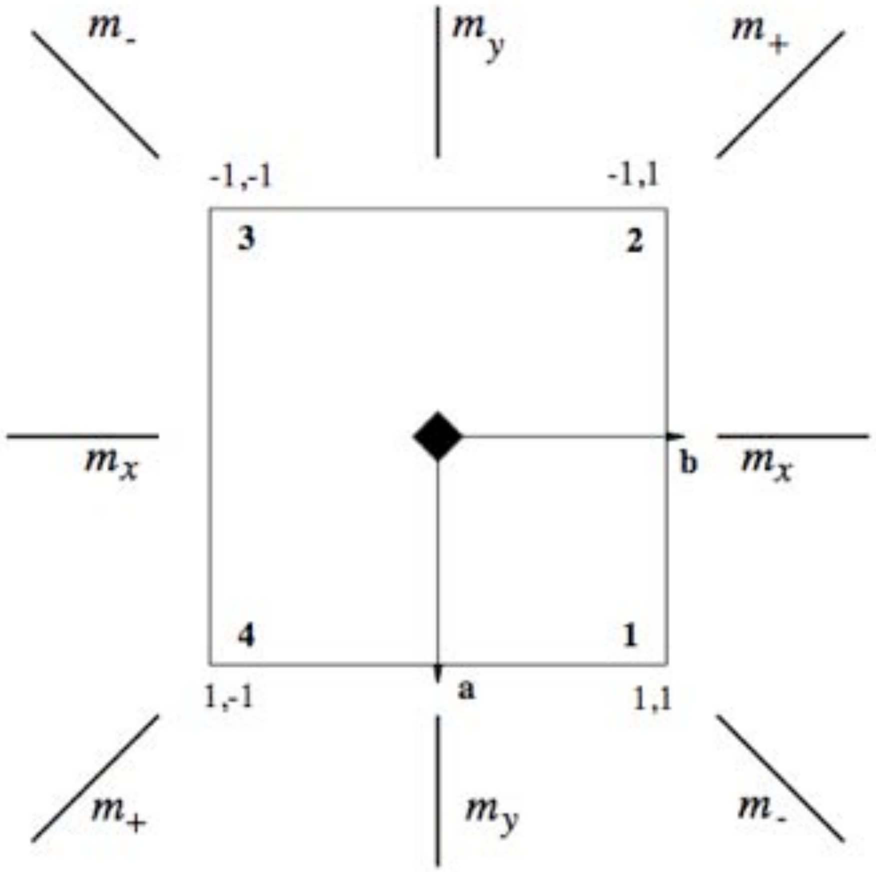
(ii) $|L(g_i)|$ is a divisor of $|G|$ (iii) $L(e)=\{e\}$

(iv) if $g_i, g_j \in L$, then $(g_i)^k=(g_j)^k= e$

Example (Problem 2.8):

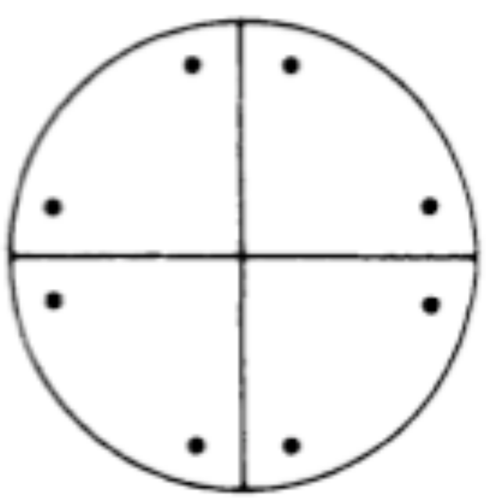
Classes of conjugate elements

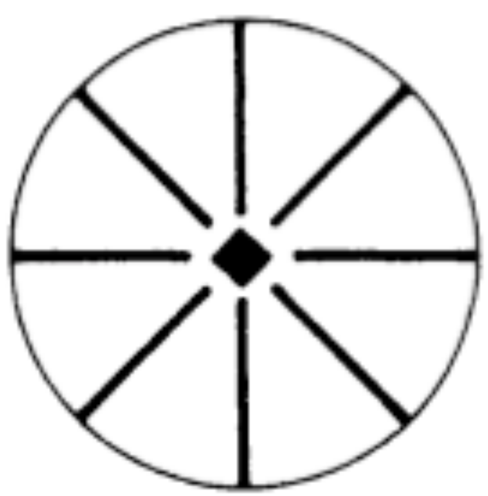
The group of the square ***4mm***



	1	2	4	4^{-1}	m_x	m_+	m_y	m_-
1	1	2	4	4^{-1}	m_x	m_+	m_y	m_-
2	2	1	4^{-1}	4	m_y	m_-	m_x	m_+
4	4	4^{-1}	2	1	m_+	m_y	m_-	m_x
4^{-1}	4^{-1}	4	1	2	m_-	m_x	m_+	m_y
m_x	m_x	m_y	m_-	m_+	1	4^{-1}	2	4
m_+	m_+	m_-	m_x	m_y	4	1	4^{-1}	2
m_y	m_y	m_x	m_+	m_-	2	4	1	4^{-1}
m_-	m_-	m_+	m_y	m_x	4^{-1}	2	4	1

Multiplication table of $4mm$





Classes of conjugate elements:
{1}, {2},{4,4⁻},{m_x,m_y}, {m_+,m_-}

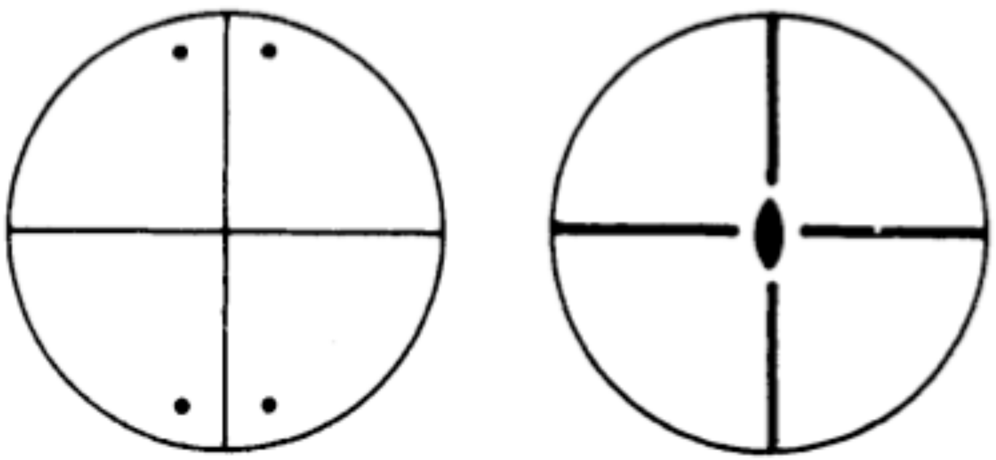
EXERCISES

Problem 2.7 (cont)

Distribute the symmetry elements of the group **mm2** = $\{1, 2_z, m_x, m_y\}$ in classes of conjugate elements.

multiplication table

×	1	2	m_x	m_y
1	1	2	m_x	m_y
2	2	1	m_y	m_x
m_x	m_x	m_y	1	2
m_y	m_y	m_x	2	1



stereographic projection

GROUP-SUBGROUP RELATIONS

I. Subgroups: index, coset decomposition and normal subgroups

II. Conjugate subgroups

III. Group-subgroup graphs

Subgroups: Some basic results (summary)

Subgroup H < G

1. $H=\{e,h_1,h_2,\ldots,h_k\} \subset G$
2. H satisfies the group axioms of G

Proper subgroups H < G, and
trivial subgroup: {e}, G

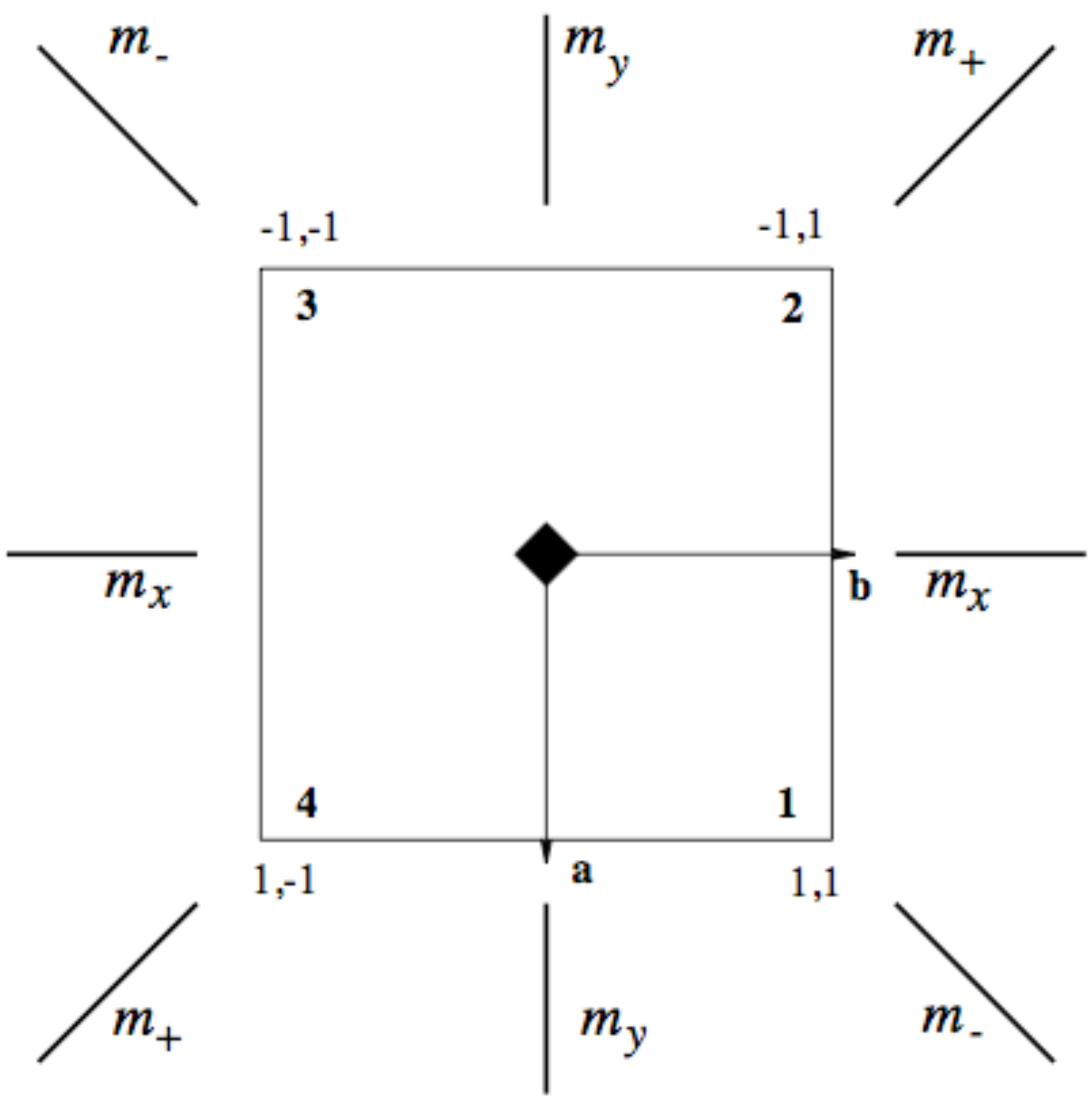
Index of the subgroup H in G: $[i]=|G|/|H|$
(order of G)/(order of H)

Maximal subgroup H of G
NO subgroup Z exists such that:
$H < Z < G$

EXERCISES

Problem 2.11

Consider the group of the square and determine its subgroups



	1	2	4	4^{-1}	m_x	m_+	m_y	m_-
1	1	2	4	4^{-1}	m_x	m_+	m_y	m_-
2	2	1	4^{-1}	4	m_y	m_-	m_x	m_+
4	4	4^{-1}	2	1	m_+	m_y	m_-	m_x
4^{-1}	4^{-1}	4	1	2	m_-	m_x	m_+	m_y
m_x	m_x	m_y	m_-	m_+	1	4^{-1}	2	4
m_+	m_+	m_-	m_x	m_y	4	1	4^{-1}	2
m_y	m_y	m_x	m_+	m_-	2	4	1	4^{-1}
m_-	m_-	m_+	m_y	m_x	4^{-1}	2	4	1

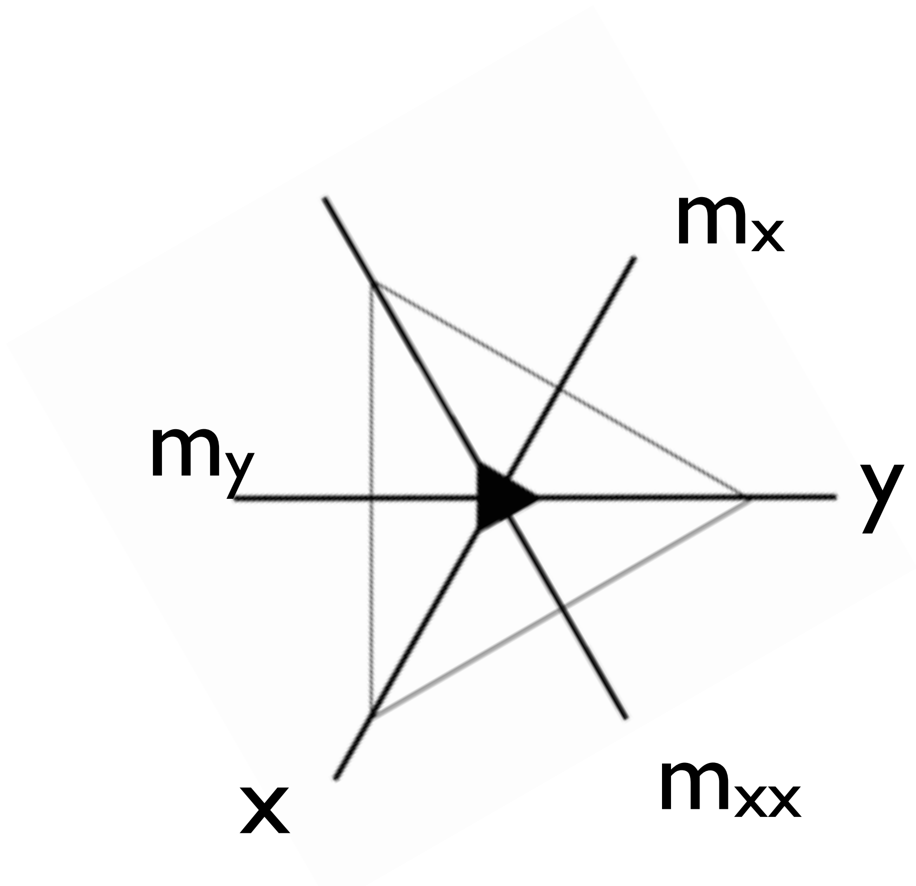
Multiplication table of $4mm$

Problem 2.12 (additional)

(i) Consider the group of the equilateral triangle and determine its subgroups;

(ii) Distribute the subgroups into classes of conjugate subgroups;

(iii) Construct the maximal subgroup graph of 3m



×	1	3^+	3^-	m_{xx}	m_x	m_y
1	1	3^+	3^-	m_{xx}	m_x	m_y
3^+	3^+	3^-	1	m_y	m_{xx}	m_x
3^-	3^-	1	3^+	m_x	m_y	m_{xx}
m_{xx}	m_{xx}	m_x	m_y	1	3^+	3^-
m_x	m_x	m_y	m_{xx}	3^-	1	3^+
m_y	m_y	m_{xx}	m_x	3^+	3^-	1

Multiplication table of **3m**

Coset decomposition G:H

Group-subgroup pair $H < G$

left coset decomposition

$G = H + g_2H + \ldots + g_mH,\ g_i \notin H,$
$m = \text{index of } H \text{ in } G$

right coset decomposition

$G = H + Hg_2 + \ldots + Hg_m,\ g_i \notin H$
$m = \text{index of } H \text{ in } G$

Coset decomposition-properties

(i) $g_iH \cap g_jH = \{\varnothing\}$, if $g_i \notin g_jH$

(ii) $|g_iH| = |H|$

(iii) $g_iH = g_jH,\ g_i \in g_jH$

Coset decomposition G:H

Normal subgroups

$Hg_j = g_jH$, for all $g_j = 1, \ldots, [i]$

Theorem of Lagrange

group G of order |G|
subgroup H<G of order |H|

then

|H| is a divisor of |G|
and [i]=|G:H|

Corollary

The order *k* of any element of G, $g^k = e$, is a divisor of |G|

EXERCISES

Problem 2.13

Consider the subgroup {e,2} of 4mm, of index 4:

-Write down and compare the right and left coset decompositions of 4mm with respect to {e,2};

-Are the right and left coset decompositions of 4mm with respect to {e,2} equal or different? Can you comment why?

Problem 2.14

Demonstrate that H is always a normal subgroup if $|G:H|=2$.

Conjugate subgroups

Conjugate subgroups Let $H_1 < G$, $H_2 < G$

then, $H_1 \sim H_2$, if $\exists\, g \in G$: $g^{-1} H_1 g = H_2$

(i) Classes of conjugate subgroups: $L(H)$

(ii) If $H_1 \sim H_2$, then $H_1 \cong H_2$

(iii) $|L(H)|$ is a divisor of $|G|/|H|$

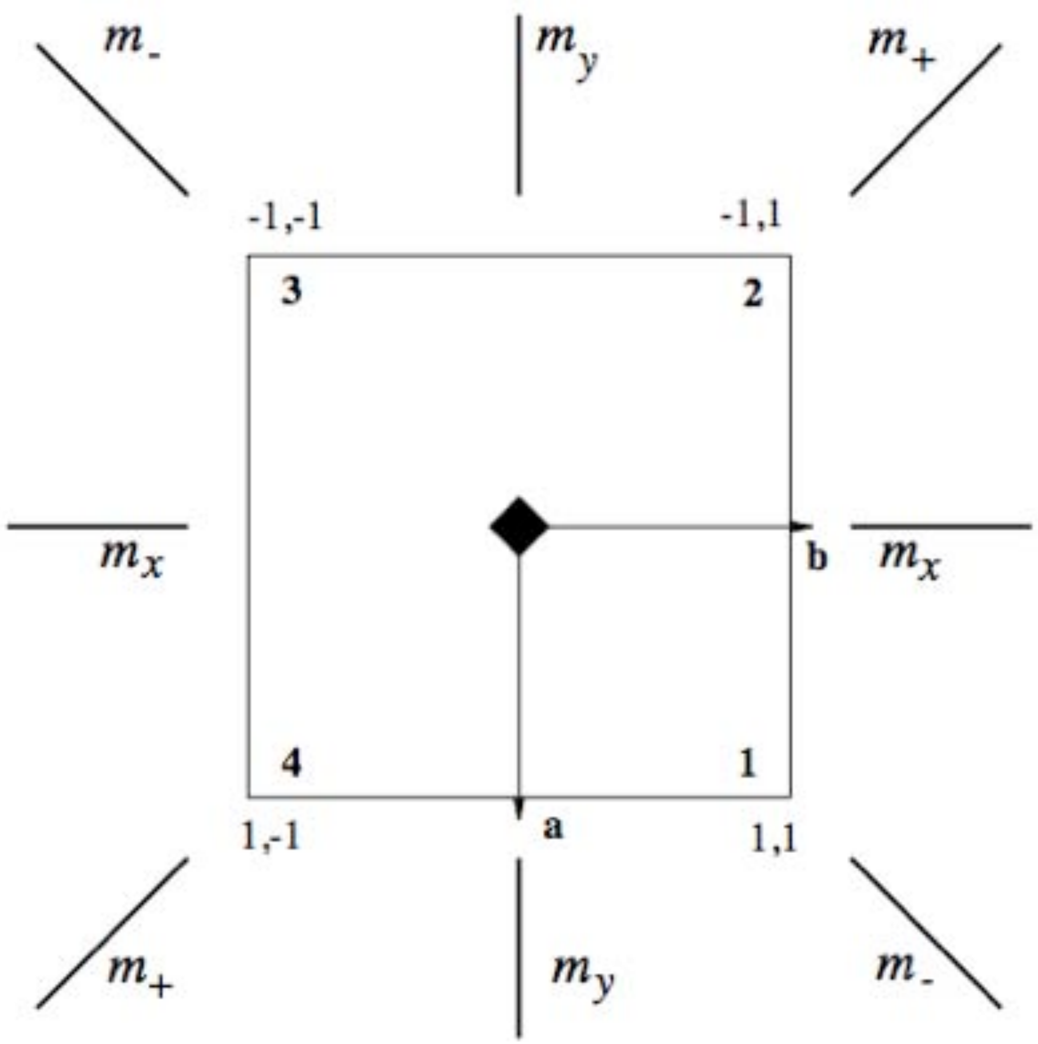
Normal subgroup

$H \triangleleft G$, if $g^{-1} H g = H$, for $\forall g \in G$

EXERCISES

Problem 2.11 (cont)

Distribute the subgroups of the group of the square into classes of conjugate subgroups

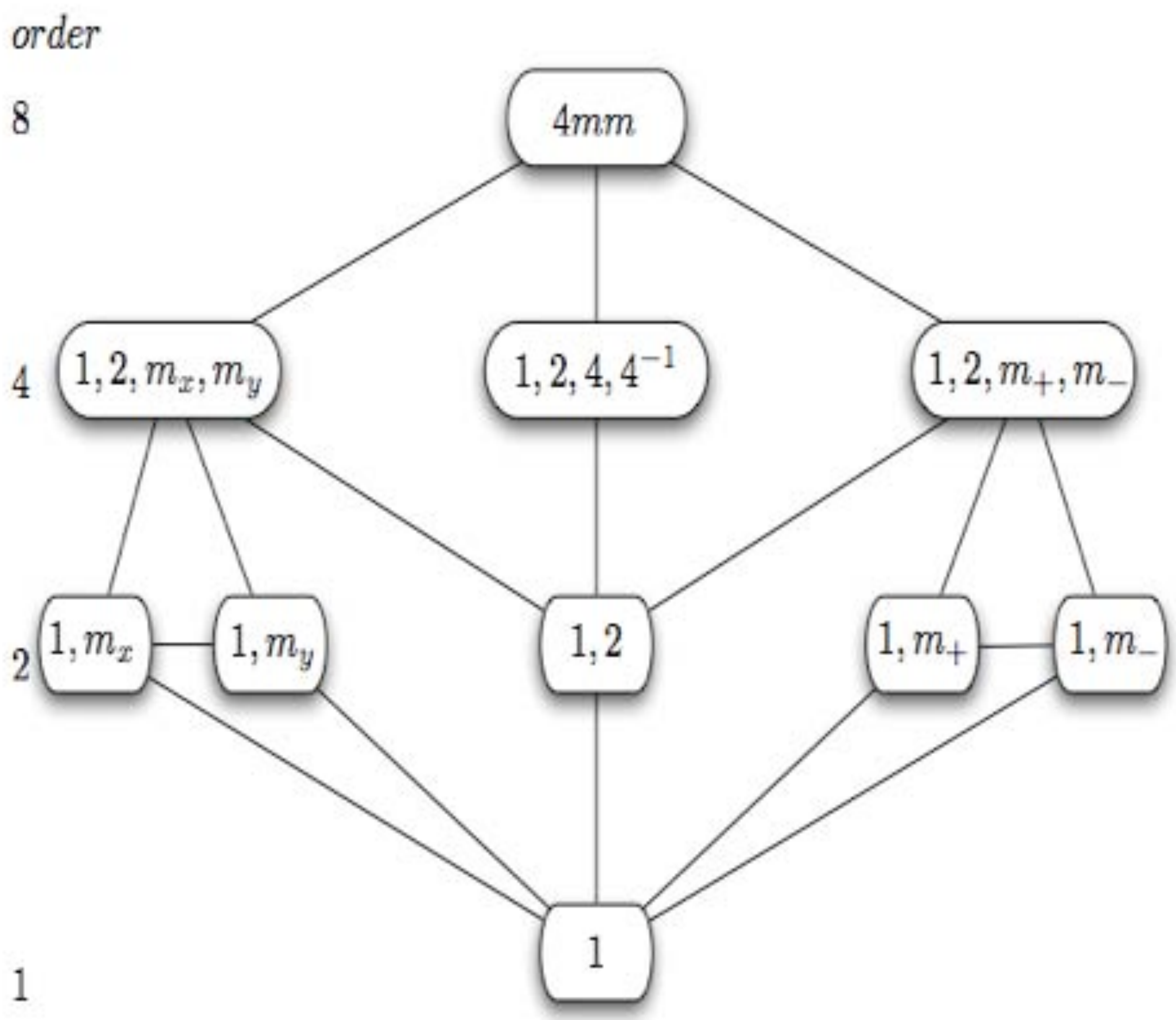


	1	2	4	4^{-1}	m_x	m_+	m_y	m_-
1	1	2	4	4^{-1}	m_x	m_+	m_y	m_-
2	2	1	4^{-1}	4	m_y	m_-	m_x	m_+
4	4	4^{-1}	2	1	m_+	m_y	m_-	m_x
4^{-1}	4^{-1}	4	1	2	m_-	m_x	m_+	m_y
m_x	m_x	m_y	m_-	m_+	1	4^{-1}	2	4
m_+	m_+	m_-	m_x	m_y	4	1	4^{-1}	2
m_y	m_y	m_x	m_+	m_-	2	4	1	4^{-1}
m_-	m_-	m_+	m_y	m_x	4^{-1}	2	4	1

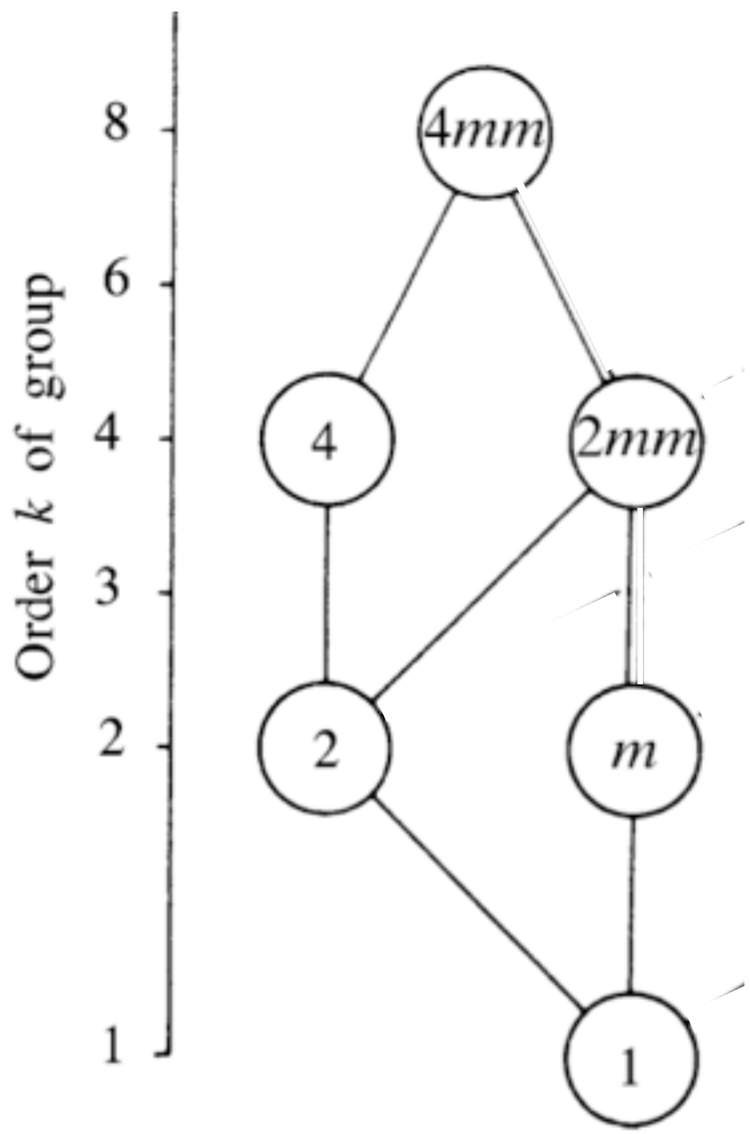
Multiplication table of $4mm$

Hint: The stereographic projections could be rather helpful

Complete and contracted group-subgroup graphs



Complete graph of maximal subgroups



Contracted graph of maximal subgroups

Group-subgroup relations of point groups

Hahn and Klapper

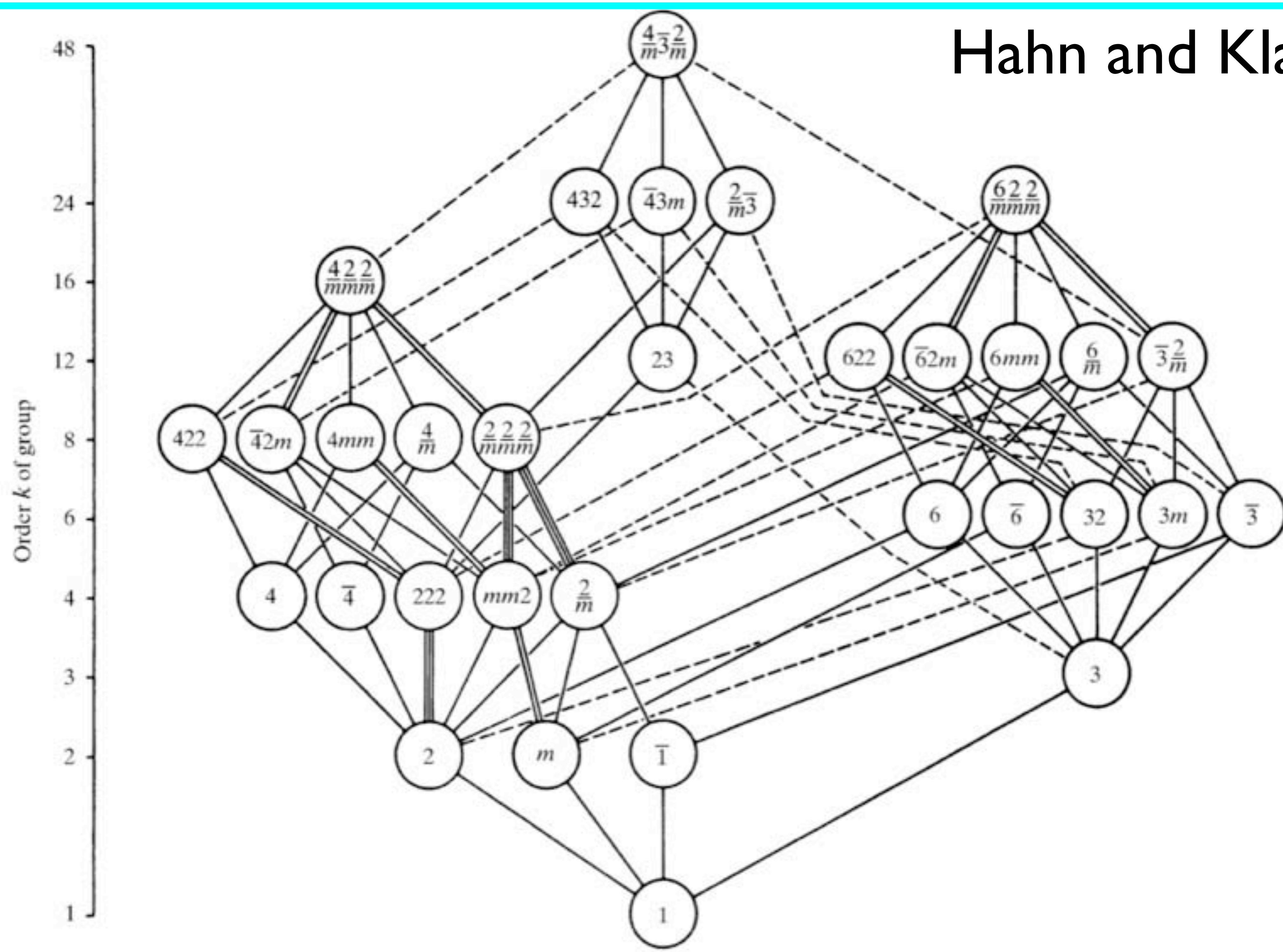


Fig. 10.1.3.2. Maximal subgroups and minimal supergroups of the three-dimensional crystallographic point groups. Solid lines indicate maximal normal subgroups; double or triple solid lines mean that there are two or three maximal normal subgroups with the same symbol. Dashed lines refer to sets of maximal conjugate subgroups. The group orders are given on the left. Full Hermann–Mauguin symbols are used.

GENERAL AND SPECIAL WYCKOFF POSITIONS

General and special Wyckoff positions

Site-symmetry group S_o={W} of a point X_o

$$WX_o = X_o$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

General position X_o

$$S = \mathbf{1} = \{1\}$$

Special position X_o

$$S > \mathbf{1} = \{1, ...,\}$$

Site-symmetry groups: oriented symbols

Example

General and special Wyckoff positions

Point group **2** = $\{1, 2_z\}$

Site-symmetry group $S_o = \{W\}$ of a point $X_o = (0,0,z)$

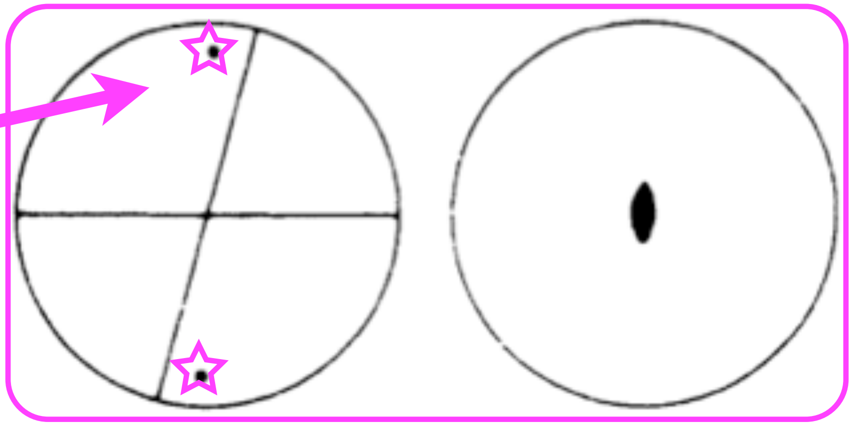
S_o = **2**

$WX_o = X_o$

$$2_z: \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$$

2 b 1 (x,y,z) (-x,-y,z)

1 a 2 (0,0,z)



Example

General and special Wyckoff positions

Point group **mm2** = $\{1, 2_z, m_x, m_y\}$

Site-symmetry group $S_o = \{W\}$ of a point $X_o = (0,0,0)$

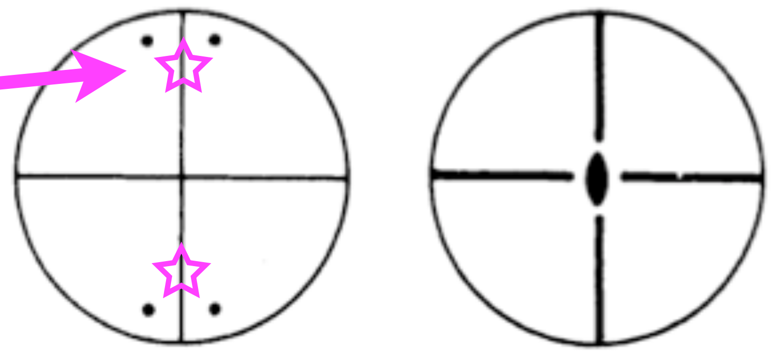
S_o = **mm2**

$WX_o = X_o$

$$2_z: \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$$

$$m_y: \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$$

4 d	1	(x,y,z) (-x,-y,z) (x,-y,z) (-x,y,z)
2 c	m..	(0,y,z) (0,-y,z)
2 b	.m.	(x,0,z) (-x,0,z)
1 a	mm2	(0,0,z)



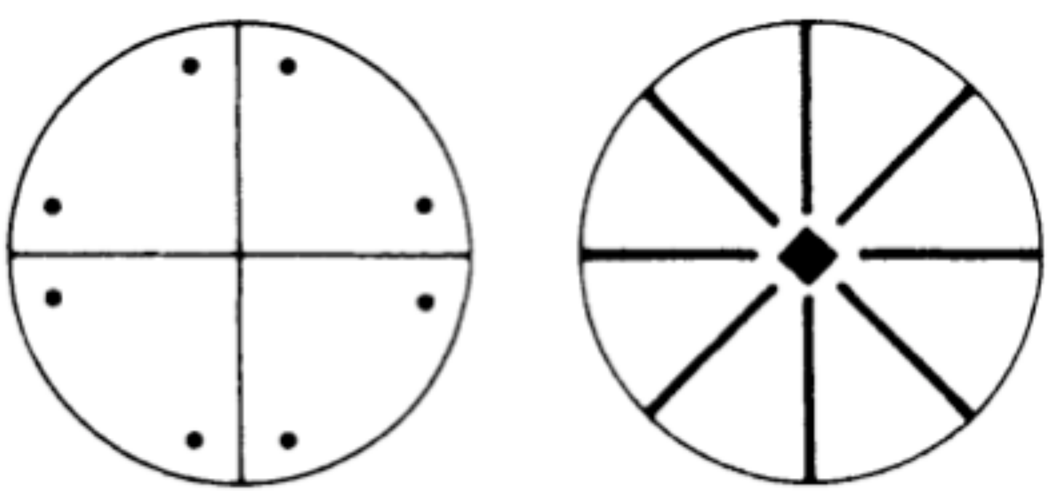
EXERCISES

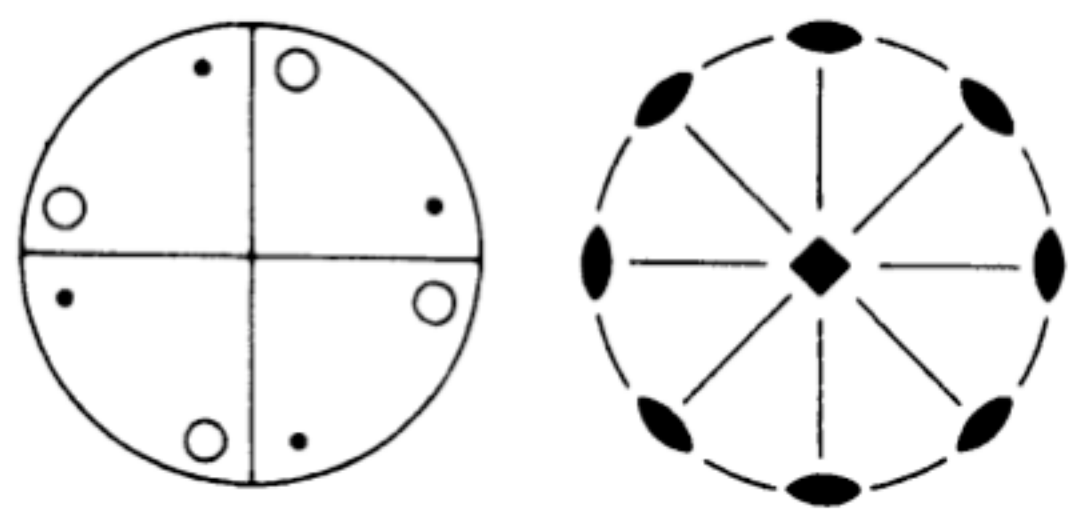
Problem 2.16

Consider the symmetry group of the square **4mm** and the point group **422** that is isomorphic to it.

Determine the general and special Wyckoff positions of the two groups.

Hint: The stereographic projections could be rather helpful



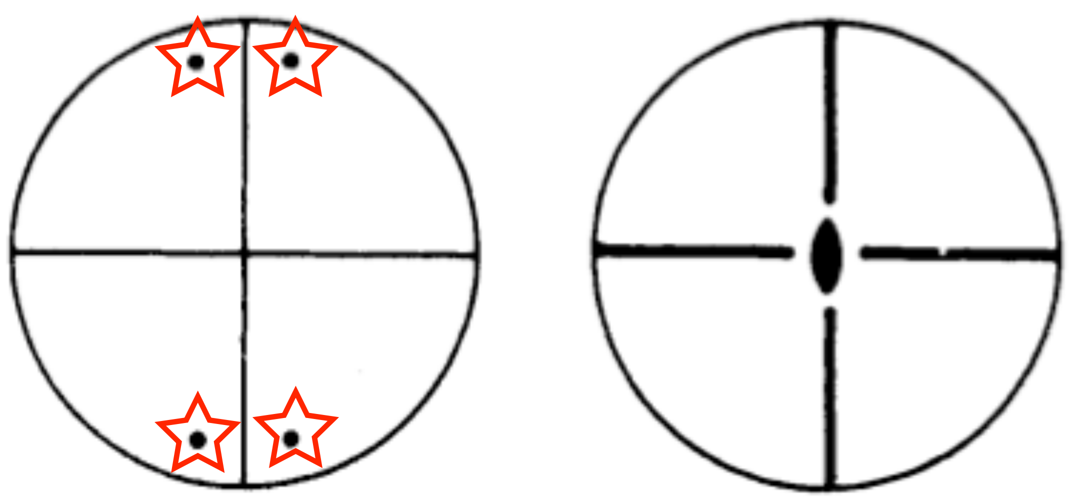


EXAMPLE

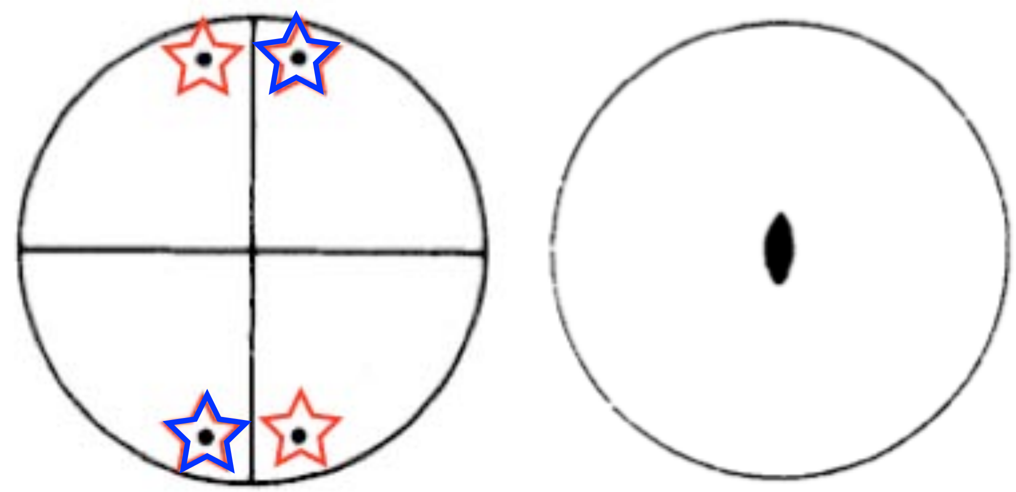
Wyckoff positions splitting schemes

Group-subgroup pair *mm2* >2, [i]=2

mm2



2



4 d 1 (x,y,z) (-x,-y,z) (x,-y,z) (-x,y,z)

$x,y,z = x_1,y_1,z_1$ 2 b 1
$-x,-y,z = -x_1,-y_1,z_1$

$x,-y,z = x_2,y_2,z_2$ 2 b 1
$-x,y,z = -x_2,-y_2,z_2$

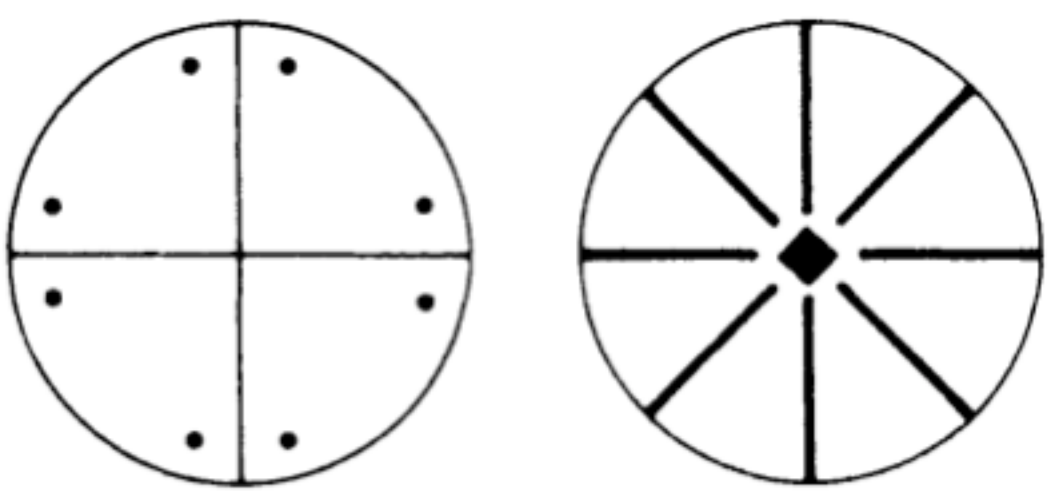
EXERCISES

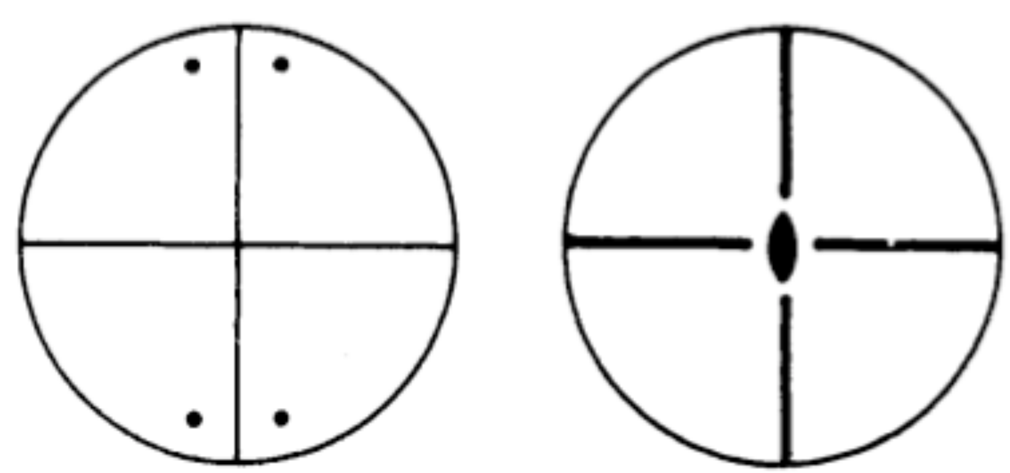
Problem 2.17

Consider the general and special Wyckoff positions of the symmetry group of the square **4mm** and those of its subgroup **mm2** of index 2.

Determine the splitting schemes of the general and special Wyckoff positions for **4mm** > **mm2**.

Hint: The stereographic projections could be rather helpful





NORMALIZERS

Normalizer of H in G

Normal subgroup

$$H \triangleleft G, \text{ if } g^{-1} H g = H, \text{ for } \forall g \in G$$

Normalizer of H in G, H<G

$$N_G(H) = \{g \in G, \text{ if } g^{-1} H g = H\}$$

$$G \geq N_G(H) \geq H$$

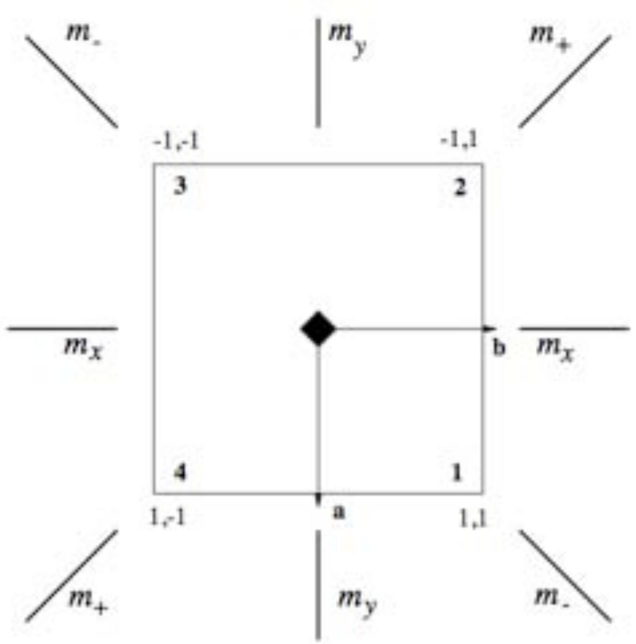
What is the normalizer $N_G(H)$ if $H \triangleleft G$?

Number of subgroups $H_i < G$ in a conjugate class

$$n = [G : N_G(H)]$$

Problem 2.18

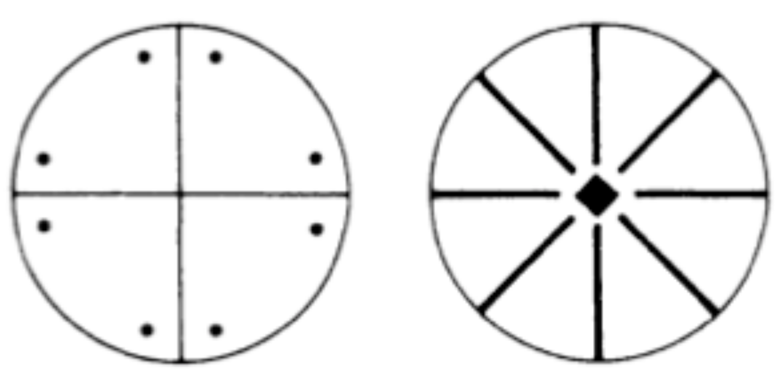
Consider the group 4*mm* and its subgroups of index 4. Determine their **normalizers** in 4*mm*. Distribute the subgroups into conjugacy classes with the help of their normalizers in 4*mm*.



	1	2	4	4^{-1}	m_x	m_+	m_y	m_-
1	1	2	4	4^{-1}	m_x	m_+	m_y	m_-
2	2	1	4^{-1}	4	m_y	m_-	m_x	m_+
4	4	4^{-1}	2	1	m_+	m_y	m_-	m_x
4^{-1}	4^{-1}	4	1	2	m_-	m_x	m_+	m_y
m_x	m_x	m_y	m_-	m_+	1	4^{-1}	2	4
m_+	m_+	m_-	m_x	m_y	4	1	4^{-1}	2
m_y	m_y	m_x	m_+	m_-	2	4	1	4^{-1}
m_-	m_-	m_+	m_y	m_x	4^{-1}	2	4	1

Multiplication table of 4*mm*

Hint: The stereographic projections could be rather helpful



ADDITIONAL

GROUP-SUPERGROUP RELATIONS

Supergroups: Some basic results (summary)

Supergroup G>H

$H=\{e,h_1,h_2,...,h_k\} \subset G$

Proper supergroups G>H, and
trivial supergroup: H

Index of the group H in supergroup G: $[i]=|G|/|H|$
(order of G)/(order of H)

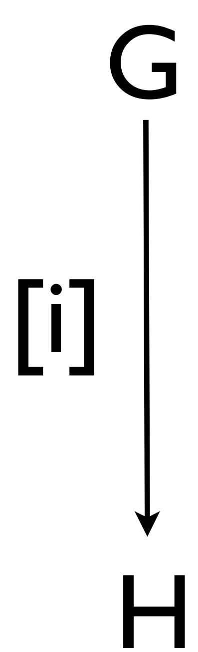
Minimal supergroups G of H

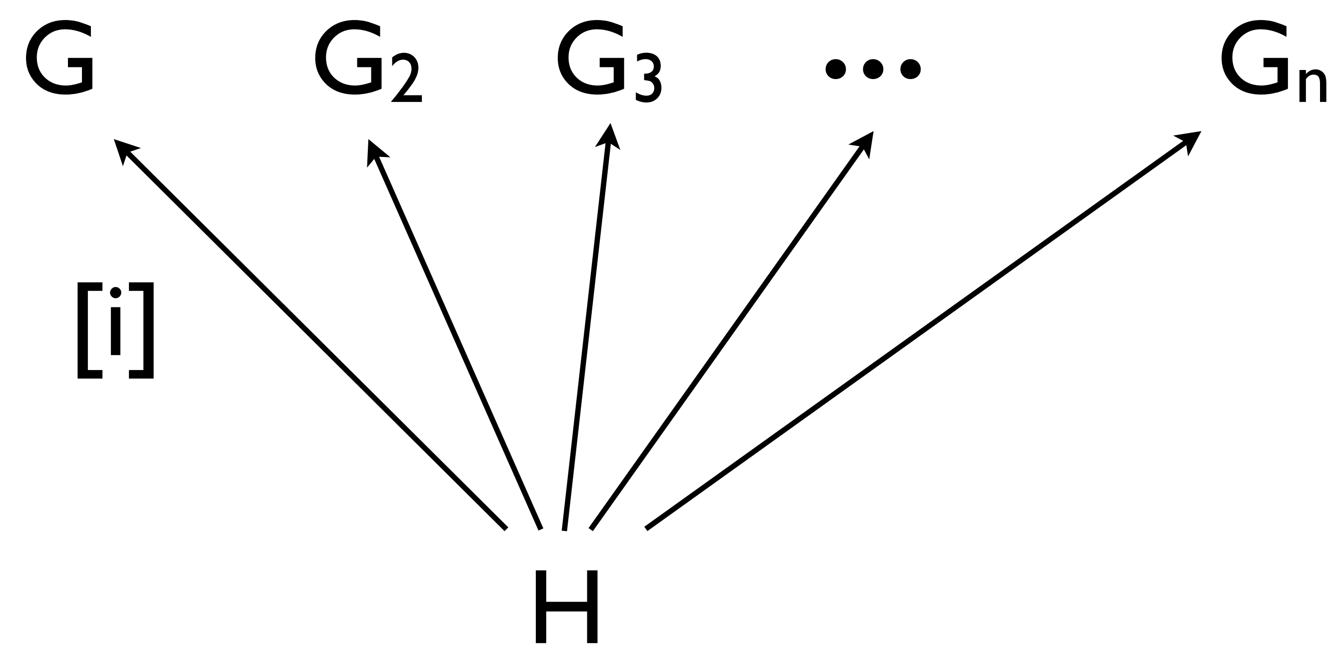
NO subgroup Z exists such that:
$H < Z < G$

The Supergroup Problem

Given a group-subgroup pair G>H of index [i]

Determine: all $G_k>H$ of index [i], $G_i \simeq G$



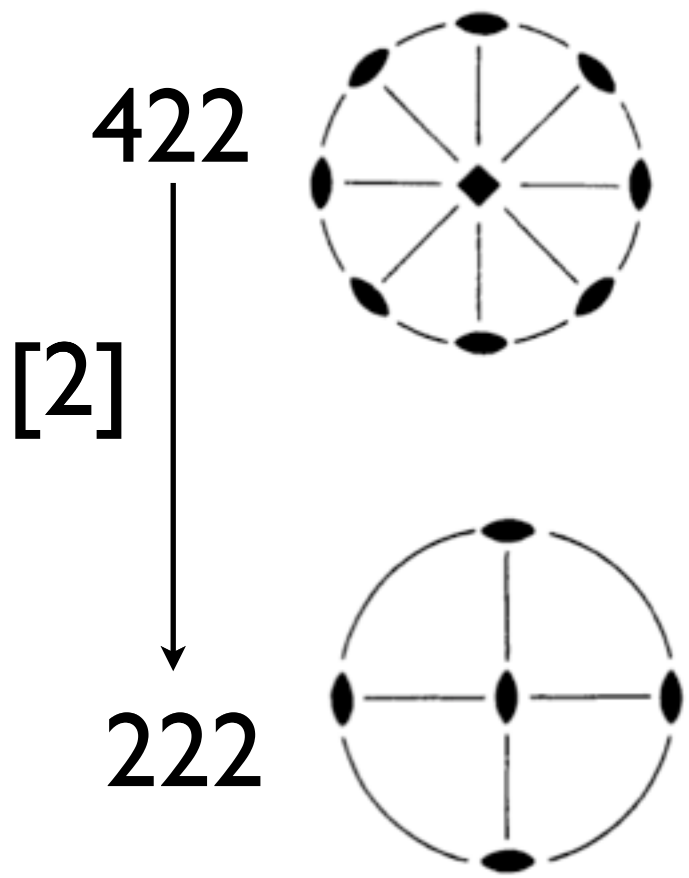


all $G_k>H$ contain H as subgroup

$$G_k=H+g_2H+...+g_{ik}H$$

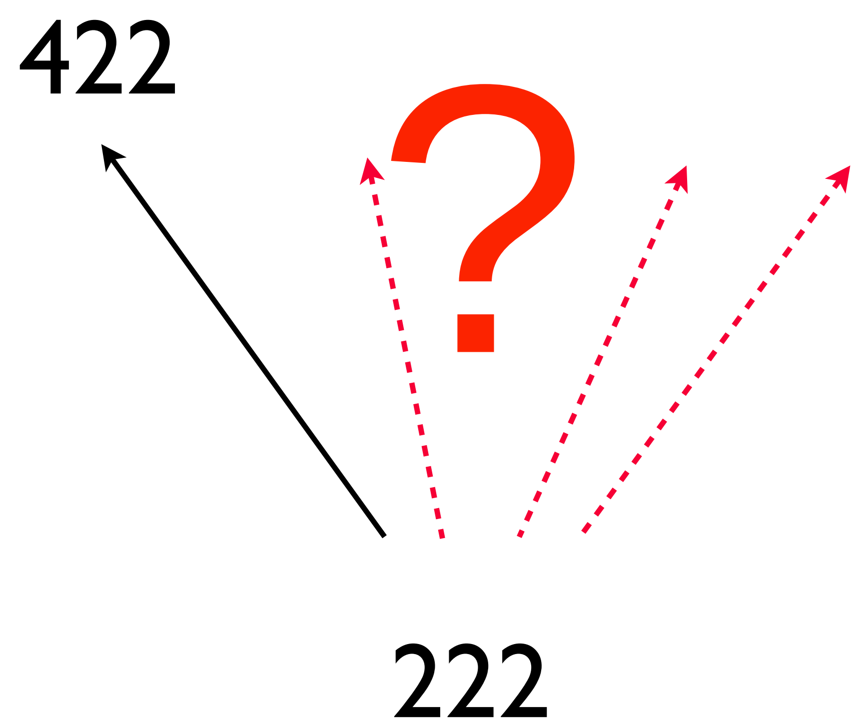
Example: Supergroup problem

Group-subgroup pair 422>222



How many are the subgroups 222 of 422?

Supergroups 422 of the group 222



How many are the supergroups 422 of 222?

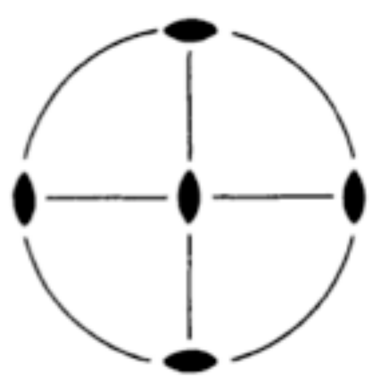
Example: Supergroup problem

Group-subgroup pair 422>222

422

[2]

$2_z2_x2_y$ $\quad$ $2_z2_+2_-$



$$4_z22 = 2_z2_x2_y + 4_z(2_z2_x2_y)$$

$$4_z22 = 2_z2_+2_- + 4_z(2_z2_+2_-)$$

Supergroups 422 of the group 222

4_z22 $\quad$ 4_x22 $\quad$ 4_y22

[2]

222

$$4_z22 = 222 + 4_z222$$

$$4_y22 = 222 + 4_y222$$

$$4_x22 = 222 + 4_x222$$

GENERATION OF CRYSTALLOGRAPHIC POINT GROUPS

Generation of point groups

Crystallographic groups are **solvable** groups

Composition series: $1 \lhd Z_2 \lhd Z_3 \lhd \ldots \lhd G$

index 2 or 3

Set of generators of a group is a set of group elements such that each element of the group can be obtained as an ordered product of the generators

$$W = (g_h)^{k_h} * (g_{h-1})^{k_{h-1}} * \ldots * (g_2)^{k_2} * g_1$$

g_1 - identity

$g_2, g_3, \ldots$ - generate the rest of elements

Example Generation of the group of the square

	2_z		4_z		m_x	
Composition series: $\boldsymbol{1}$	$\lhd$	$\boldsymbol{2}$	$\lhd$	$\boldsymbol{4}$	$\lhd$	$\boldsymbol{4mm}$
	[2]		[2]		[2]	

Step 1:

$\boldsymbol{1} = \{1\}$

Step 2:

$\boldsymbol{2} = \{1\} + 2_z \{1\}$

Step 3:

$\boldsymbol{4} = \{1,2\} + 4_z \{1,2\}$

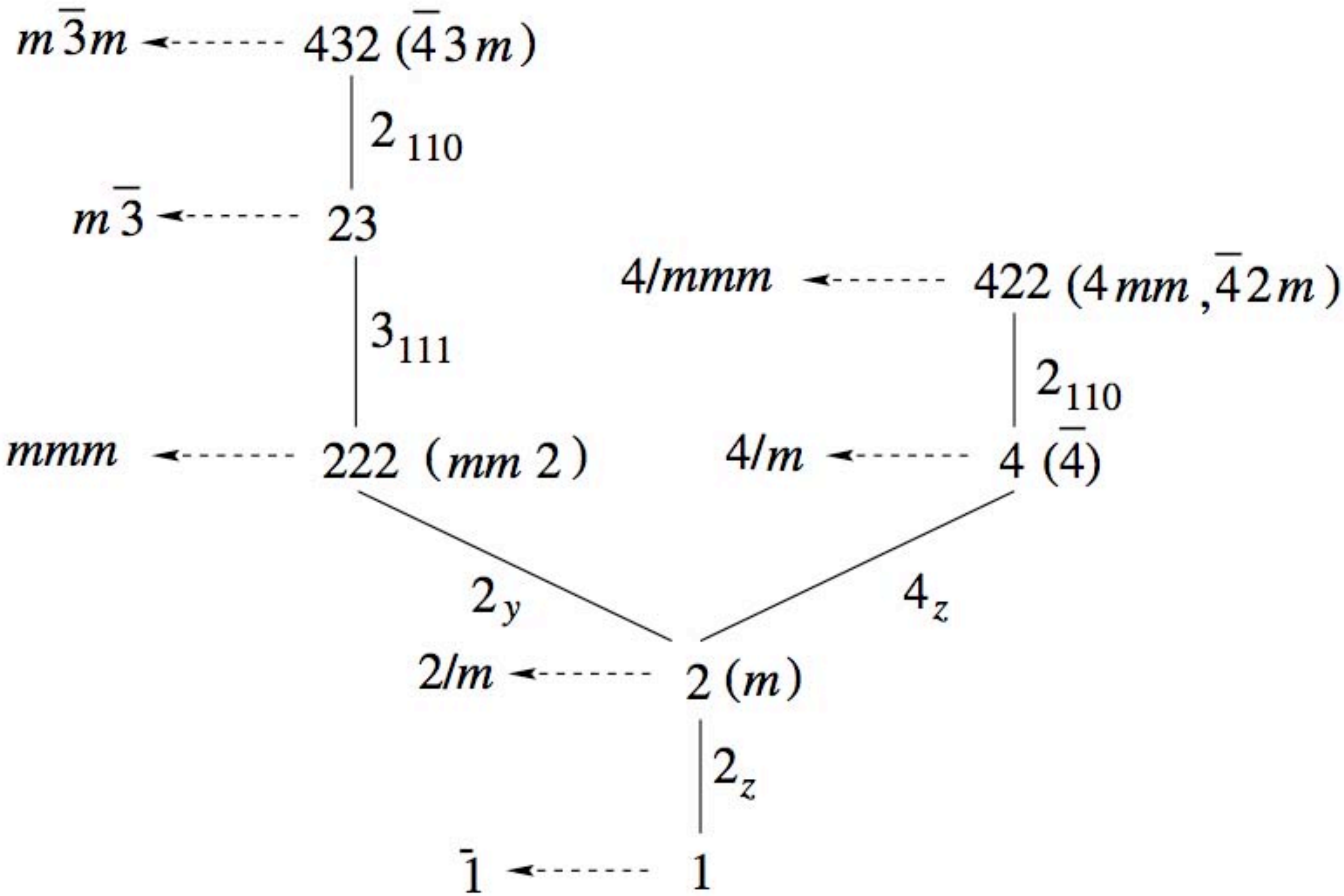
Step 4:

$\boldsymbol{4mm} = \boldsymbol{4} + m_x \boldsymbol{4}$

	1	2	4	4^{-1}	m_x	m_+	m_y	m_-
1	1	2	4	4^{-1}	m_x	m_+	m_y	m_-
2	2	1	4^{-1}	4	m_y	m_-	m_x	m_+
4	4	4^{-1}	2	1	m_+	m_y	m_-	m_x
4^{-1}	4^{-1}	4	1	2	m_-	m_x	m_+	m_y
m_x	m_x	m_y	m_-	m_+	1	4^{-1}	2	4
m_+	m_+	m_-	m_x	m_y	4	1	4^{-1}	2
m_y	m_y	m_x	m_+	m_-	2	4	1	4^{-1}
m_-	m_-	m_+	m_y	m_x	4^{-1}	2	4	1

Multiplication table of $4mm$

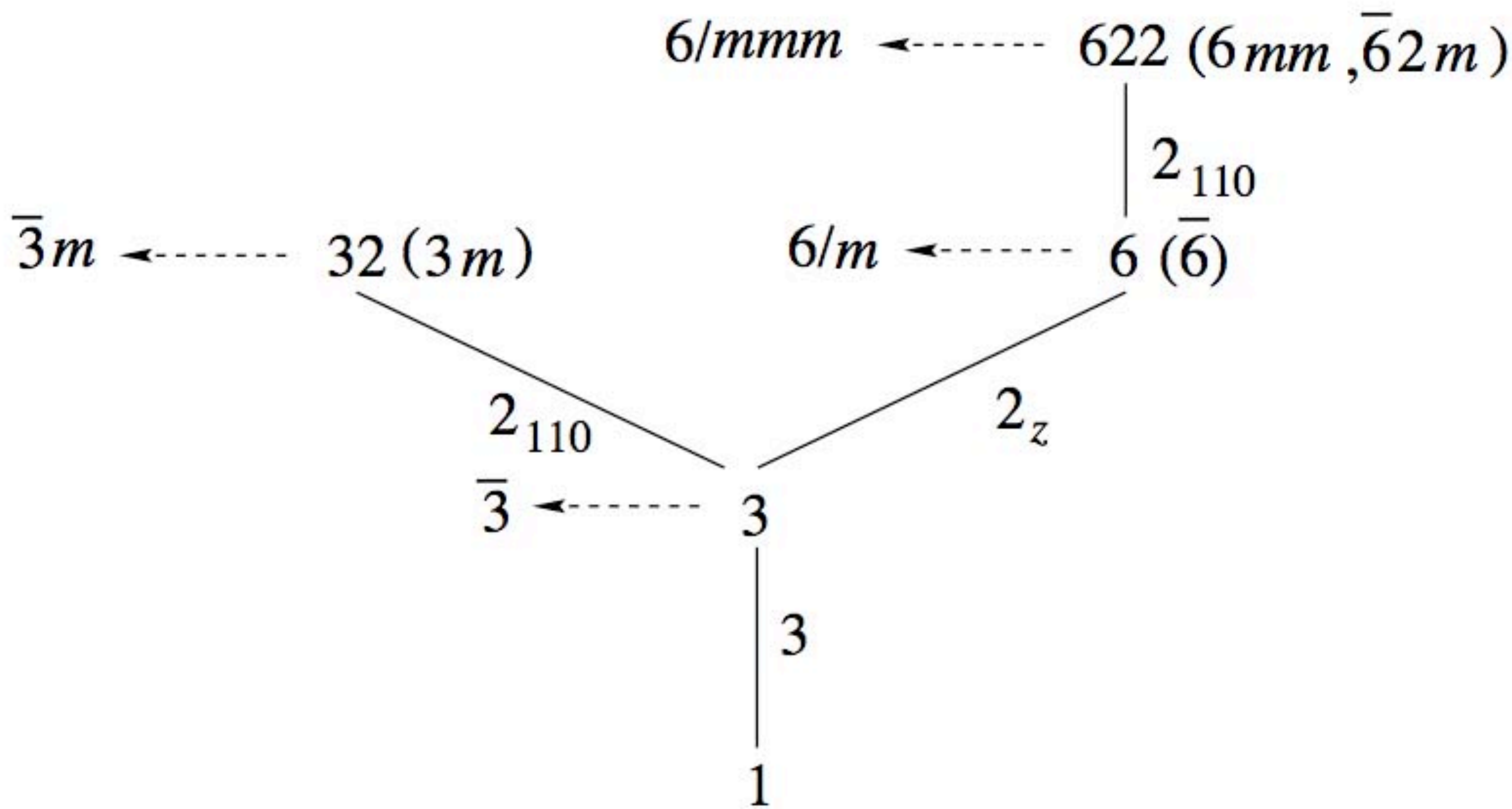
Generation of sub-cubic point groups



Composition series of cubic point groups and their subgroups

HM Symbol	SchoeSy	generators	compos. series
1	$\mathcal{C}_1$	1	1
$\bar{1}$	$\mathcal{C}_i$	$1, \bar{1}$	$\bar{1} \triangleright 1$
2	$\mathcal{C}_2$	$1, 2$	$2 \triangleright 1$
m	$\mathcal{C}_s$	$1, m$	$m \triangleright 1$
$2/m$	$\mathcal{C}_{2h}$	$1, 2, \bar{1}$	$2/m \triangleright 2 \triangleright 1$
222	$\mathcal{D}_2$	$1, 2_z, 2_y$	$222 \triangleright 2 \triangleright 1$
$mm2$	$\mathcal{C}_{2v}$	$1, 2_z, m_y$	$mm2 \triangleright 2 \triangleright 1$
mmm	$\mathcal{D}_{2h}$	$1, 2_z, 2_y, \bar{1}$	$mmm \triangleright 222 \triangleright \ldots$
4	$\mathcal{C}_4$	$1, 2_z, 4$	$4 \triangleright 2 \triangleright 1$
$\bar{4}$	$\mathcal{S}_4$	$1, 2_z, \bar{4}$	$\bar{4} \triangleright 2 \triangleright 1$
$4/m$	$\mathcal{C}_{4h}$	$1, 2_z, 4, \bar{1}$	$4/m \triangleright 4 \triangleright \ldots$
422	$\mathcal{D}_4$	$1, 2_z, 4, 2_y$	$422 \triangleright 4 \triangleright \ldots$
$4mm$	$\mathcal{C}_{4v}$	$1, 2_z, 4, m_y$	$4mm \triangleright 4 \triangleright \ldots$
$\bar{4}2m$	$\mathcal{D}_{2d}$	$1, 2_z, \bar{4}, 2_y$	$\bar{4}2m \triangleright \bar{4} \triangleright \ldots$
$4/mmm$	$\mathcal{D}_{4h}$	$1, 2_z, 4, 2_y, \bar{1}$	$4/mmm \triangleright 422 \triangleright \ldots$
23	$\mathcal{T}$	$1, 2_z, 2_y, 3_{111}$	$23 \triangleright 222 \triangleright \ldots$
$m\bar{3}$	$\mathcal{T}_h$	$1, 2_z, 2_y, 3_{111}, \bar{1}$	$m\bar{3} \triangleright 23 \triangleright \ldots$
432	$\mathcal{O}$	$1, 2_z, 2_y, 3_{111}, 2_{110}$	$432 \triangleright 23 \triangleright \ldots$
$\bar{4}3m$	$\mathcal{T}_d$	$1, 2_z, 2_y, 3_{111}, m_{1\bar{1}0}$	$\bar{4}3m \triangleright 23 \triangleright \ldots$
$m\bar{3}m$	$\mathcal{O}_h$	$1, 2_z, 2_y, 3_{111}, 2_{110}, \bar{1}$	$m\bar{3}m \triangleright 432 \triangleright \ldots$

Generation of sub-hexagonal point groups



Composition series of hexagonal point groups and their subgroups

HM Symbol	SchoeSy	generators	compos. series
1	$\mathcal{C}_1$	1	1
3	$\mathcal{C}_3$	$1,\, 3$	$3 \triangleright 1$
$\overline{3}$	$\mathcal{S}_6$	$1,\, 3,\, \overline{1}$	$\overline{3} \triangleright 3 \triangleright 1$
32	$\mathcal{D}_3$	$1,\, 3,\, 2_{110}$	$32 \triangleright 3 \triangleright 1$
$3m$	$\mathcal{C}_{3v}$	$1,\, 3,\, m_{110}$	$3m \triangleright 3 \triangleright 1$
$\overline{3}m$	$\mathcal{D}_{3d}$	$1,\, 3,\, 2_{110},\, \overline{1}$	$\overline{3}m \triangleright 32 \triangleright \ldots$
6	$\mathcal{C}_6$	$1,\, 3,\, 2_z$	$6 \triangleright 3 \triangleright 1$
$\overline{6}$	$\mathcal{C}_{3h}$	$1,\, 3,\, m_z$	$\overline{6} \triangleright 3 \triangleright 1$
$6/m$	$\mathcal{C}_{6h}$	$1,\, 2,\, 2_z,\, \overline{1}$	$6/m \triangleright 6 \triangleright \ldots$
622	$\mathcal{D}_6$	$1,\, 3,\, 2_z,\, 2_{110}$	$622 \triangleright 6 \triangleright \ldots$
$6mm$	$\mathcal{C}_{6v}$	$1,\, 3,\, 2_z,\, m_{110}$	$6mm \triangleright 6 \triangleright \ldots$
$\overline{6}2m$	$\mathcal{D}_{3h}$	$1,\, 3,\, m_z,\, 2_{110}$	$\overline{6}2m \triangleright \overline{6} \triangleright \ldots$
$6/mmm$	$\mathcal{D}_{6h}$	$1,\, 3,\, 2_z,\, 2_{110},\, \overline{1}$	$6/mmm \triangleright 622 \triangleright \ldots$

Problem 2.15

Generate the symmetry operations of the group *4/mmm* following its composition series.

Generate the symmetry operations of the group $\bar{3}m$ following its composition series.